Journal of the Korean
Statistical Society
Vol. 21, No. 2, 1992

Scaling Limits for Associated Random Measures†

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ABSTRACT

In this paper we investigate scaling limits for associated random measures satisfying some moment conditions. No stationarity is required. Our results imply an improvement of a central limit theorem of Cox and Grimmett to associated random measure and an extension to the nonstationary case of scaling limits of Burton and Waymire. Also we prove an invariance principle for associated random measures which is an extension of the Birkel's invariance principle for associated processes.

KEYWORDS: Association, random measure, central limit theorem, invariance principle, cluster random measure.

1. INTRODUCTION

Many recent papers have been concerned with various limit theorems for associated random elements. Newman[10] has established the central limit theorem for stationary families of random variables indexed by Z^d, the set of all d-tuples of integers ($d \geq 1$, a positive integer), under simple summability decay rate condition on the correlations by exploiting an often natural additional condition of association. Also Newman and Wright[11, 12] have improved it to an invariance principle for

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† This research is supported by Korea Science and Engineering Foundation Grant under KOSEF 911-0105-017-1.

stationary sequences of associated random variables under the same conditions for $d = 1, 2$. Analogous results hold in the case of random measures. For example Burton and Waymire[4] have proved scaling limits for stationary associated random measures which are an extension of Newman's ideas in the case of nonlattice random fields. Cox and Grimmett[5] have proved a central limit theorem for a family of associated random variables indexed by the lattice Z^d by replacing the stationary property with conditions on the moments of the random variables ; this is an extension of the Newman-Wright's invariance principle for associated stationary sequence $\{X_n : n \geq 1\}$ satisfying $\text{Cov}(X_1, X_1) + 2\sum_{j=2}^{\infty} \text{Cov}(X_1, X_j) < \infty$. Moreover, Birkel[2] has investigated an invariance principle for nonstationary associated processes ; this result implies an improvement of a central limit theorem of Cox and Grimmett(1984) and an extension of an invariance principle of Newman and Wright(1981).

In this paper we investigate Cox and Grimmett's idea in the case of nonlattice random field by the similar method to that of Burton and Waymire[4] and prove an invariance principle for nonstationary associated random measure in dimension one. Preliminary definitions and previous results are given in Section 2. An extension of Burton and Waymire's scaling limits to the nonstationary case and an invariance principle for nonstationary associated random measure in dimension one are derived in Section 3. Finally we apply this scaling limit to Poisson center cluster random measures in Section 4.

2. PRELIMINARIES

Let $\mathcal{B}^d$ denote the collection of Borel subsets of d-dimensional Euclidean space R^d. The set M of all nonnegative measures μ defined on $(R^d, \mathcal{B}^d)$ and finite on bounded sets(i.e.,Radon measures) will be equipped with the smallest sigma field $\mathcal{M}$ containing basic sets of the form $\{\mu \in M : \mu(A) \leq r\}$ for $A \in \mathcal{B}^d, 0 \leq r \leq \infty$.

Definition 2.1. A random measure X is a measurable map from a probability space $(\Omega, \mathcal{F}, P)$ to $(M, \mathcal{M})$. In the special case when the distribution of X is concentrated on the class N of nonnegative integer valued Radon measures we refer to X as a point random field(point process). M becomes a Polish space when equipped with the vague topology and the sigma field $\mathcal{M}$ coincides with the Borel sigma field for this tolopogy (cf.,Kallenberg, 1983). Moreover N is closed in the vague tolopogy for M (and therefore measurable). We shall denote the restriction of the sigma field $\mathcal{M}$ to N by $\mathcal{N}$. A random measure $X(\cdot)$ may also be defined by specifying the finite dimensional distributions of the random variables $X(B)$ where, for different choice of the bounded Borel subset B and R^d, $X(B)$ represents the mass of the random measure X. A point random field may alternately be considered as an nonnegative integer valued random measure and the most well known random measure is the Poisson random measure with parameter ρ. X has this distribution if whenever $B_1, \cdots, B_n$ are disjoint bounded Borel sets then $X(B_1), \cdots, X(B_n)$ are independent Poisson random variables with respective parameters $\rho|B_1|, \cdots, \rho|B_n|$ where $|\ |$ denotes Lebesgue measure. Another example is a Gaussian random measure which is a mean zero Gaussian process $X(A)$, indexed by set A in a σ-field, such that $X(\cup A_i) = \sum X(A_i)$, where A_i's are disjoint and the series on the right is required to converge everywhere [8]. M has a partial ordering defined by $\mu \leq \nu$ if for each bounded Borel set $B, \mu(B) \leq \nu(B)$ (See Kallenberg(1983) for more complete discussion of random measures).

Recall that an infinite family $\mathcal{G}$ of random variables is associated if for every finite subfamily $\{Y_1, \cdots, Y_n\} \subset \mathcal{G}$ and for every pair of coordinatewise increasing functions f, g on R^n, $\text{Cov}(f(Y_1, \cdots, Y_n), g(Y_1, \cdots, Y_n)) \geq 0$ (See Esary, Proschan, and Walkup(1967)). Burton and Waymire [4] extended this notion to random measure.

Definition 2.2.(Burton, Waymire, 1985) A random measure X is associated if whenever $F, G : M \to R$ is measurable and increasing with respect to the partial ordering on M then $\text{Cov}[F(X), G(X)]$ is nonegative whenever the covariance is defined. It follows from works of Burton and Kim[3], Burton and Waymire[4] that X is associated if and only if the family of random variables $\{X(B) : B$ bounded Borel set $\}$ is associated.

Definition 2.3. (Burton,Waymire,1985) If X is a random measure we define the λ-renormalization of X to be the signed random measure X where $X_\lambda(B) = \lambda^{-d/2}[X(\lambda B) - EX(\lambda B)]$. We consider X as a random element of $D[0,1]^d$ by setting $X_\lambda(t) = X_\lambda([0,t])$ where $[0,t]$ is the rectangle $[0,t_1] \times \ldots \times [0,t_d]$.

Definition 2.4. (Burton, Waymire, 1985) Let X be a random measure. X satisfies a classical scaling limit if for all disjoint rectangles $A_1, \cdots, A_n$,

$$([X(\lambda A_1) - E(X(\lambda A_1))]/\lambda^{d/2}, \cdots, [X(\lambda A_n) - E(X(\lambda A_n))]/\lambda^{d/2})$$

converges in distribution(as $\lambda \to \infty$) to a multivariate normal with mean vector 0 and diagonal covariance matrix whose diagonal terms are $\sigma^2|A_1|, \cdots, \sigma^2|A_n|$ for some positive parameter σ^2, where $|A_i|$ equals the Lebesgue measure of A_i.

Newman[10] proved that the renormalized block sums of stationary associated random variables indexed by Z^d converge in distribution to independent Gaussian random variables if the covariance function is summable. Burton and Waymire[4] extended this notion to associated random measure as follows.

Theorem 2.5.(Burton, Waymire, 1985) Suppose that X is a stationary associated point random measure with $EX(I) = 0, EX^2(I) < \infty$ and X satisfies

$$0 < \sum_{\underline{k} \in Z^d} \text{Cov}(X(I), X(I + \underline{k})) = \sigma^2 < \infty, \tag{2.1}$$

where $I = (0, 1]^d$ is the unit cube. Then

(1) X satisfies a classical scaling limit with parameter σ^2.

(2) X fulfills an invariance principle for $d = 1, 2$.

Moreover the assertion remains true if X is a stationary associated(finite additive) random interval function satisfying (2.1).
Note that whether an invariance principle for associated random fields for $d > 2$ is still open[9].
Before discussing Cox and Grimmett's central limit theorem we introduce some notation. If $\underline{x} \in Z^d$, we write x_i for the ith coordinate of $\underline{x}$. For $\underline{x}, \underline{y} \in Z^d$, we write $\underline{x} \leq \underline{y}$ (respectively $\underline{x} < \underline{y}$) if $x_i \leq y_i$ (respectively $x_i < y_i$) for all i. We define $|\underline{x} - \underline{y}| = \sup\{|x_i - y_i| : i = 1, 2, \ldots, d\}$ and write $\underline{1}$ for a vector with unit entries. If $\underline{k} \in Z^d$, we denote the box $\{\underline{x} \in Z^d : \underline{1} \leq \underline{x} \leq \underline{k}\}$ by $\Lambda(\underline{k})$.

Cox and Grimmett(1984) weakened the assumption of strict stationarity and replaced it by certain conditions on the moments of the random variables. They also used the coefficient

$$u(r) = \sup_{\underline{k} \in Z^d} \sum_{\underline{j}: |\underline{j} - \underline{k}| \geq r} \text{Cov}(X_{\underline{j}}, X_{\underline{k}}), r \in N \cup \{0\} \quad \underline{j}, \underline{k} \in Z^d \tag{2.2}$$

which is the covariance structure and obtained the following central limit theorem.

Theorem 2.6. (Cox, Grimmett, 1984) Suppose that for each $n \in N$, $\{X_{\underline{j}} : \underline{j} \in \Lambda(n\underline{1})\}$ is a family of associated random variables with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. Assume

$$u(r) \xrightarrow{r} 0, \quad u(0) < \infty, \tag{2.3}$$

$$\inf_{\underline{j}\in\Lambda(n\underline{1})} \operatorname{Var}(X_{\underline{j}}) > 0, \quad n \in N, \tag{2.4}$$

$$\sup_{\underline{j}\in\Lambda(n\underline{1})} E(|X_{\underline{j}}|^3) < \infty, \quad n \in N, \tag{2.5}$$

Then $\{X_{\underline{j}} : \underline{j} \in \Lambda(n\underline{1})\}$ satisfies the central limit theorem.
For $d = 1, \underline{1} = 1$ and the $X_{\underline{j}}$'s stationary, this is the central limit theorem of Newman and Wright(1981), but subject to a superfluous third moment condition.

3.RESULTS

Cox and Grimmett[5] have proved a cental limit theorem for nonstationary associated random variables indexed by the lattice Z^d as in Theorem 2.6. We extend this theorem to associated random measures(i.e. the case of nonlattice random fields) as follows: First we define the coefficient which is a covariance structure for $\underline{i}, \underline{j} \in Z^d$

$$v(r) = \sup_{\underline{i}} \sum_{\underline{j}:|\underline{j}-\underline{i}|\geq r} \operatorname{Cov}(X(I+\underline{j})), X(I+\underline{i}), r \in N \cup \{0\}, \tag{3.1}$$

where I is the unit interval. And using the coefficient (3.1) we obtain the following central limit throrem.

Theorem 3.1. Let X be an associated random measure with $E(X^2(I+\underline{j})) < \infty$. Assume

$$v(r) \xrightarrow{r} 0, v(0) < \infty. \tag{3.2}$$

$$\inf_{\underline{j}\in Z^d} \operatorname{Var} X(I+\underline{j}) > 0, \tag{3.3}$$

$$\sup_{\underline{j}\in Z^d} E(|X(I+\underline{j})|^3) < \infty, \tag{3.4}$$

where I is the unit interval. Then X satisfies a central limit theorem.

Proof. Let $I = (0,1]^d$ and let X denote a random interval function(a family of random variables $X(J)$ indexed by the family I of all finite intervals J, see page 317,[6]) subject to the conditions of the theorem. Put $X_{\underline{j}} = X(I+\underline{j}-\underline{1}) - EX(I+\underline{j}-\underline{1})$. Then for $n \in N$, $\{X_{\underline{j}} : \underline{j} \in \Lambda(n\underline{1})\}$ is a family of associated random variables with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$ and satisfies conditions (2.3), (2.4) and (2.5) in Theorem 2.6 and hence $\{X_{\underline{j}} : \underline{j} \in \Lambda(n\underline{1})\}$ satisfies a central limit theorem. Let $D = \{\underline{j} \in Z^d : \underline{j} \in \Lambda([\lambda]I)\} = \{\underline{j} \in Z^d : \underline{j} \in \Lambda([\lambda]\underline{1})\}$ where $[\lambda]$ denotes the greatest integer in λ.

Then

$$\lambda^{-d/2}[X(\lambda I) - EX(\lambda I)] = \lambda^{-d/2}[X(I^0) - EX(I^0) + \sum_{\underline{j}\in D} X_{\underline{j}}], \quad (3.5)$$

where, $I^0 = \lambda I \backslash [\lambda]I$.
Since $I^0 \subset ([\lambda]+1)I\backslash[\lambda]I$ and X is associated we obtain

$$\begin{aligned} \mathrm{Var}X(I^0) &\le \mathrm{Var}X(([\lambda]+1)I - [\lambda]I) \\ &= \mathrm{Cov}\Big(\sum_{([\lambda]-1)\underline{1}<\underline{j}\le[\lambda]\underline{1}} X(I+\underline{j}), \sum_{([\lambda]-1)\underline{1}<\underline{j}\le[\lambda]\underline{1}} X(I+\underline{j})\Big) \\ &\le ([\lambda]^d - ([\lambda]-1)^d) \sup_{\underline{k}\in Z^d} \sum_{\underline{j}:|\underline{k}-\underline{j}|\ge 0} \mathrm{Cov}(X(I+\underline{j}), X(I+\underline{k})) \\ &= ([\lambda]^d - ([\lambda]-1)^d)v(0) \\ &= 0(\lambda^{d-1}) \text{ as } \lambda \to \infty \end{aligned}$$

and hence by Chebyshev's inequality $\lambda^{-d/2}[X(I^0) - EX(I^0)]$ converges in probability to zero as $\lambda \to \infty$. Thus since $[\lambda]^d \sim \lambda^d$ as $\lambda \to \infty$, X satisfies the central limit theorem.

Remark. Note that if we put $X_{\underline{j}} = X(I+\underline{j}-\underline{1}) - EX(I+\underline{j}-\underline{1})$ and $\lambda \in N$ then Theorem 3.1 coincides with Theorem 2.6.

Theorem 3.2. Suppose that X is an associated point random measure and satisfies the conditions in Theorem 3.1. Then X satisfies a classical scaling limit.

Proof. Let $I = (0,1]^d$. By Theorem 3.1 $X_\lambda(I)$ converges in law to normal as $\lambda \to \infty$. For arbitrary disjoint unit intervals $I_1, \cdots, I_m$, the same consideration may be applied to the random vector $(X_\lambda(I_1), \cdots, X_\lambda(I_m))$ and the result follows.

Note that for a wide sense stationary associated random measure conition (2.1) implies that (3.2) and (3.3) are automatically satisfied. Therefore in the stationary case Theorem 3.3 is scaling limit of Theorem 2.5 except the superfluous third moment condition (3.4).

Next we strengthen result of Theorem 3.1 to get a functional scaling limit in dimension one by an application of results in Birkel(1988).

Let $\{X_j : j \in N\}$ be a sequence of random variables on some probability space $(\Omega, \mathcal{F}, P)$ with $EX_j = 0, EX_j^2 < \infty$. For $n \in N$ put

$$S_n = \sum_{j=1}^{n} X_j, \sigma_n^2 = ES_n^2, \text{ and define } W_n(t) = \sigma_n^{-1} S_{[nt]}, t \in [0,1]$$

where $S_0 = 0$. Then W_n is a measurable map from $(\Omega, \mathcal{F})$ into $(D, \mathcal{B}(D))$, where D is the set of all function on $[0,1]$ which have left hand limits and are continuous from the right, and $\mathcal{B}(D)$ is the Borel- σ-algebra induced by the Skorohod topology. $\{X_j : j \in N\}$ fulfills the invariance principle if W_n converges weakly to standard Browian motion W on D.

Theorem 3.3.(Birkel,1988) Let $\{X_j : j \in N\}$ be a sequence of associated random variables with $EX_j = 0, EX_j^2 < \infty$. If $\sigma_n^{-2} E(S_{nk} S_{nl}) \to \min\{k,l\}$ for $k,l \in N$ and $\{X_j : j \in N\}$ satisfies the central limit theorem then $\{X_j : j \in N\}$ fulfills the invariance principle.

For $n \in N$, put $U_n = \sum_{j=1}^{n} [X(I+j-1) - EX(I+j-1)], u_n^2 = EU_n^2$, where $U_o = 0$, and define the rescaled random measure by $X_\lambda(t) = \lambda^{-1/2}[X((0,\lambda t]) - EX((0,\lambda t])]$.

Theorem 3.4. Let X be an associated random measure with $EX^2(I+j-1) < \infty$. Assume

$$n^{-1} u_n^2 \xrightarrow{n} u^2 \in (0,\infty) \tag{3.6}$$

$$u_n^{-2} E(U_{nk}, U_{nl}) \to \min\{k,l\} \text{ for } k,l \in N, \tag{3.7}$$

$$v(r) \xrightarrow{n} 0, v(0) < \infty, r \in N \cup \{0\}, \tag{3.8}$$

$$\inf_{j \in N} \text{Var}(X(I+j-1)) > 0, \tag{3.9}$$

$$\sup_{j \in N} E[|X(I+j-1|^3] < \infty \ , \tag{3.10}$$

where $I = (0,1]$. Then X_λ converges for the Skorokhod topology on the appropriate function space to Brownian motion as $\lambda \to \infty$.

Proof. Let X denote a random interval function subject to the conditions of the theorem and consider the distribution of

$$\lambda^{-1/2}[X(\lambda t I)] - EX(\lambda t I)] \text{ as } \lambda \to \infty. \tag{3.11}$$

For $j \in N$, put $X_j = X(I+j-1) - EX(I+j-1)$, then $\{X_j\}$ satisfies the central limit theorem by conditions (3.8),(3.9), and (3.10)(see Theorem 3.1) and we have

$$\lambda^{-1/2}[X(\lambda tI) - EX(\lambda tI)] = \lambda^{-1/2}[X(I^0) - EX(I^0)] + \lambda^{-1/2}\sum_{j=1}^{[\lambda t]} X_j \tag{3.12}$$

where $I^0 = \lambda tI \backslash [\lambda t]I$. Since $I^o \subset (([\lambda t]+1)I - [\lambda t]I)$ by association we obtain

$$\mathrm{Var}X(I^0) \leq \mathrm{Var}X(([\lambda t]+1)I - [\lambda t]I) = \mathrm{Var}X(I+[\lambda t]) = EX^2(I+[\lambda t]) < \infty$$

and hence by Chebyshev's inequality the first term in the right of (3.12), $\lambda^{-1/2}[X(I^0) - EX(I^0)]$ converges in probability to zero as $\lambda \to \infty$. Next the second term in the right (3.12) yields

$$\lambda^{-1/2}\sum_{j=1}^{[\lambda_t]} X_j = (\frac{u_{[\lambda]}^2}{[\lambda]})^{\frac{1}{2}}(\frac{[\lambda]}{\lambda})^{\frac{1}{2}}(u_{[\lambda]}^{-1})\sum_{j=1}^{[\lambda t]} X_j.$$

Since $[\lambda]^{-1/2}u_{[\lambda]} \to u$ as $\lambda \to \infty$ from (3.6) and $[\lambda]/\lambda \to 1$ as $\lambda \to \infty$, $\lambda^{-1/2}\sum_{j=1}^{[\lambda t]} X_j$ converges weakly to Brownian motion as $\lambda \to \infty$ by Theorem 3.3. Thus by Theorem 4.1 in Bilingsley(1968) X_λ converges for the Skorokhod topology on the appropriate function space to Brownian motion as $\lambda \to \infty$

Remark. Note that if we put $X_j = X(I+j-1) - EX(I+j-1)$ and $\lambda \in N$, then Theorem 3.4 coincides with Theorem 3.3.

4. CLUSTER RANDOM MEASURES

In this section we apply Theorem 3.2 to Poisson center cluster random measures. These have been used as models of infinite divisibility and self-similarity as well as models of natural phenomena such as storm systems and galaxies (Burton, Kim(1988) and Burton, Waymire(1985)). These are constructed as follows. Let U be a Poisson point random field (which is not necessarily stationary) with parameter ρ . Let $V = \{V_{\underline{x}} : \underline{x} \in R^d\}$ be a collection of iid random measures with $E[V_{\underline{x}}(R^d)] = \gamma < \infty$. Then we say that X is a cluster process with centers U and members V if

$$X(B) = \sum_{\underline{x}:U(\underline{x})>0} V_{\underline{x}}(B - \underline{x})$$

for each bounded Borel set B. We denote X by $[U, V]$. It is natural to hope that moment conditions on V will imply moment conditions on X regardless of the shape B in R^d(see Burton and Kim (1988)).

Theorem 4.1. Let $X = [U, V]$ as above. Assume

$$\mathrm{E}[V_{\underline{x}}(I+\underline{j})] = 0,\ \mathrm{E}[V_{\underline{x}}(R^d)^2] = \xi < \infty, \tag{4.1}$$

$$\sup_{\underline{j}\in Z^d} \sum_{\underline{k}:|\underline{j}-\underline{k}|\geq r} \mathrm{Cov}(V_{\underline{x}}(I+\underline{j}-\underline{x}), V_{\underline{x}}(I+\underline{k}-\underline{x})) \xrightarrow{r} 0,\quad r \in N\cup\{0\}. \tag{4.2}$$

Then the coefficient (3.1), that is $v(r) = \sup_{\underline{j}\in Z^d} \sum_{\underline{k}:|\underline{j}-\underline{k}|\geq r} \mathrm{Cov}(X(I+\underline{j}), X(I+\underline{k})), r \in N \cup \{0\}$ satisfies the conditions $v(0) < \infty$ and $v(r) \xrightarrow{r} 0$.

Proof. Like in the proof of Theorem 5.3 of Burton and Waymire(1985)

$$\begin{aligned}
v(0) &= \sup_{\underline{j}\in Z^d} \sum_{\underline{k}\in Z^d} \mathrm{Cov}(X(I+\underline{j}), X(I+\underline{k})) \\
&= \sup_{\underline{j}\in Z^d} \sum_{\underline{k}\in Z^d} \int_{R^d} \mathrm{E}[V_{\underline{x}}(I+\underline{j}-\underline{x})V_{\underline{x}}(I+\underline{k}-\underline{x})]\rho d\underline{x} \\
&= \sup_{\underline{j}\in Z^d} \int_{R^d} \sum_{\underline{k}\in Z^d} \mathrm{E}[V_{\underline{x}}(I+\underline{j}-\underline{x})V_{\underline{x}}(I+\underline{k}-\underline{x})]\rho d\underline{x} \\
&= \sup_{\underline{j}\in Z^d} \int_{R^d} \mathrm{E}[V_{\underline{x}}(I+\underline{j}-\underline{x})V_{\underline{x}}(R^d)]\rho d\underline{x} \\
&\leq \rho \mathrm{E}[V_{\underline{x}}(R^d)^2] = \rho\xi < \infty, \quad \text{by assumption(4.1)}. \\
v(r) &= \sup_{\underline{j}\in Z^d} \sum_{\underline{k}:|\underline{j}-\underline{k}|\geq r} \mathrm{Cov}(X(I+\underline{j}), X(I+\underline{k})), r \in N \cup \{0\} \\
&= \sup_{\underline{j}\in Z^d} \sum_{\underline{k}:|\underline{j}-\underline{k}|\geq r} \int_{R^d} \mathrm{E}\{V_{\underline{x}}(I+\underline{j}-\underline{x})V_{\underline{x}}(I+\underline{k}-\underline{x})\}\rho d\underline{x} \\
&\leq \int_{R^d} \sup_{\underline{j}\in Z^d} \sum_{\underline{k}:|\underline{j}-\underline{k}|\geq r} \mathrm{E}\{V_{\underline{x}}(I+\underline{j}-\underline{x})V_{\underline{x}}(I+\underline{k}-\underline{x})\}\rho d\underline{x}
\end{aligned}$$

where ρ is the intensity of U. Thus $v(r) \xrightarrow{r} 0$ by assumption (4.2).
The first part of the next theorem appears in Burton and Waymire(1985) and the second part is in the joint work of Burton and Kim(1988).

Theorem 4.2. Let $X = [U, V]$ as above. Then

(1) X is associated.

(2) For a rectangular box B in R^d and for $0 \leq \delta \leq 2$, there is a constant K depending only on δ and $|B|$ so that $E[|X(B)|^{2+\delta}] \leq KE[(V_{\underline{x}}(R^d))^{2+\delta}]$.

Theorem 4.3. Let $X = [U, V]$ as above and satisfy (4.1) and (4.2). Assume

$$\inf_{\underline{j}\in Z^d} \mathrm{Var} V_{\underline{x}}(I + \underline{j} - \underline{x}) > 0, \tag{4.3}$$

$$\mathrm{E}(|V_{\underline{x}}(R^d)|^3) < \infty. \tag{4.4}$$

Then X satisfies a classical scaling limit.

Proof. First note that by assumption (4.1), $EX(I + \underline{j}) = \int_{R^d} E(V_{\underline{x}}(I + \underline{j} - \underline{x}))\rho d\underline{x} = 0$, $E[X(I + \underline{j})^2] \leq \rho E[(V_{\underline{x}}(R^d))^2] < \infty$ and also by Theorem 4.1, $v(0) < \infty$ and $v(r) \xrightarrow{r} 0$. To see $\inf_{\underline{j}\in Z^d} \mathrm{Var} X(I + \underline{j}) > 0$ simply we use

$$\begin{aligned} \mathrm{Var} X(I + \underline{j}) &= \mathrm{Cov}(X(I + \underline{j}), X(I + \underline{j})) \\ &= \textstyle\int_{R^d} \mathrm{E}[V_{\underline{x}}(I + \underline{j} - \underline{x})^2]\rho d\underline{x} \end{aligned}$$

(See the proof of Theorem 5.3 of Burton and Waymire(1985))

$$= \textstyle\int_{R^d} \mathrm{Var} V_{\underline{x}}(I + \underline{j} - \underline{x})\rho d\underline{x},$$

so that

$$\begin{aligned} \inf_{\underline{j}\in Z^d} \mathrm{Var} X(I + \underline{j}) &= \inf_{\underline{j}\in Z^d} \int_{R^d} \mathrm{Var} V_{\underline{x}}(I + \underline{j} - \underline{x})\rho d\underline{x} \\ &\geq \textstyle\int_{R^d} \rho \inf_{\underline{j}\in Z^d} \mathrm{Var} V_{\underline{x}}(I + \underline{j} - \underline{x}) d\underline{x} > 0 \text{ by assumption (4.3).} \end{aligned}$$

From (2) of Theorem 4.2 we have $\sup_{\underline{j}\in Z^d} \mathrm{E}(|X(I + \underline{j})|^3) \leq K\mathrm{E}(V_{\underline{x}}(R^d))^3 < \infty$.
Thus by (1) of Theorem 4.2 and Theorem 3.2 X satisfies a classical scaling limit.

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