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The Concept of Fuzzy Probability

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ABSTRACT

Since Zadeh's definition for probability of fuzzy event is presented, alternative definitions for probability of fuzzy event is suggested. Also various properties of these new definitions have been presented. In this paper it is our purpose to show the works continued by finding a natural definition of a fuzzy probability measure on an arbitrary fuzzy measurable space. Thus, the main process is to observe fuzzy probability measure to be qualified by weak axioms of boundary condition, monotonicity and continuity suggested by Klir(1988). Especially, we will show that these axioms are satisfied through in succession of modifications from the Yager's method.

1. INTRODUCTION

In classical situation of stochastic uncertainty, an event A is a member of σ-field $\mathcal{A}$, of subsets of a sample space Ω. A probability measure P is a normalized measure over a measurable sapce$(\Omega, \mathcal{A})$, that is, P is a real valued function which assigns to every A in $\mathcal{A}$ a probability, $P(A)$ such that

(i) $P(A) \geq 0, \ A \in \mathcal{A}$

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(ii) $P(\Omega) = 1$

(iii) If $A_i \in \mathcal{A}$, $i \in I \subset N$, pairwise disjoint, then $P(\bigcup_{i\in I} A_i) = \sum_{i\in I} P(A_i)$

As defined above, an event is a precisely specified collection of points in the sample space. By contrast, we encounter situations in which an event is a fuzzy rather than a sharply defined collection of points. For example, ill defined events such as "It will be a warm day tomorrow", "In tosses of a coin, there are several more heads than tails" and "X is a small integer" are not crisply defined, that is, that the happening of these events is not certain and that we want to express the probability of its happening. These are fuzzy events in the sense that there is no sharp dividing line between its occurrence and nonoccurrence. Such an event may be characterized as a fuzzy subset of the sample space Ω, with the membership function suggested by Zadeh(1965). Also, Zadeh(1968) and Yager(1979) proposed that an event is defined on a possibility distribution(see Zadeh(1978)) as a fuzzy set, then it must be distinguished from typical probability theory.

In our view, probability of "fuzzy event" is imprecisely as fuzzy rather than crisp numbers. Such Probability which will be referred to as fuzzy probability are exemplified by the above instances which are labeled as warm, several, small.

The first attempt at precise meaning to the notions of fuzzy event and fuzzy measure was made in the paper "Probability measures of fuzzy events" by Zadeh (1968). Since the paper was presented, the concept of fuzzy probability has been discussed by Yager(1979), Klement(1982).

We begin the discussion of fuzzy probability with the distinct definitions which fuzzy probability should be a scalar and can be considered as a fuzzy set. We shall consider both views.

2. THE CONCEPT OF FUZZY EVENT

In above examples, "warm" is a fuzzy subset corresponding to a fuzzy event defined over the unit interval. If fuzzy set A shall be used to indicate a fuzzy event of X,for each $x_i \in X$, $\mu_A(x_i)$ shall be used to indicate the grade of membership of x_i in A. In this framework, attribute of fuzzy set A shall be used by using membership function μ_A of possibility distribution.

The first notion in Zadeh's work to the fuzzy event and fuzzy measure was made in a classical probability space, $(R^n, \mathcal{A}, P)$. Thus fuzzy event in R^n was defined as a measurable map $\mu_A : R^n \to [0,1]$. But there are another works by giving a natural definition of fuzzy probability measure on an arbitrary fuzzy measurable space. Fuzzy σ-algebra and fuzzy measure related to the concepts introduced by

Klement(1981). Now we define analogously fuzzy event as an event in usual probability space.

Let Ω be an arbitrary set. Let $A : \Omega \to [0,1]$. If $\mathcal{A}$ is a family of fuzzy sets on Ω with the following properties

(A1) $\Omega \in [0,1]: \ \Omega \in \mathcal{A}$

(A2) $A \in \mathcal{A}$: $1 - A \in \mathcal{A}$

(A3) $A_n \in \mathcal{A}$, $n = 1, 2, \ldots : \ \sup_n(A_n) \in \mathcal{A}$

then $\mathcal{A}$ is called a fuzzy σ-algebra on Ω. The elements of $\mathcal{A}$ are called fuzzy event or fuzzy measurable sets. Also $(\Omega, \mathcal{A})$ is called a fuzzy measurable space. Note that A is nonfuzzy or fuzzy, Zadeh's fuzzy event exists on the usual probability space, not fuzzy σ -algebra on Ω.

Given a fuzzy σ-algebra on Ω, there exists a smallest classical σ-algebra on Ω making all "function" in σ -measurable. If we assume $(\Omega, \mathcal{A})$ is a fuzzy measurable space, the fuzzy probability measure by Klement(1981) is a mapping of $\mathcal{A} \to [0,1]$ fulfilling the probabilities

(p1) $P(0) = 0, \ P(1) = 1$

(p2) if $A, B \in \mathcal{A}$, then $P(A \vee B) + P(A \wedge B) = P(A) + P(B)$,
where $P(A \vee B) = \bigcup \alpha[P(x|\mu_A(x) \geq \alpha) + P(x|\mu_B(x) \geq \alpha) - P(x|\mu_{A \wedge B}(x) \geq \alpha)]$
$P(A \wedge B) = \bigcup \alpha P(x|\mu_{A \wedge B}(x) \geq \alpha)$

(p3) if $\forall A_n \in \mathcal{A}$, $(n = 1, 2, \ldots)$, $A_1 \subset A_2 \subset, \cdots, \subset A_n$ then $P(A_1) < P(A2) <, \cdots, < P(A_n)$

The triplet $(\Omega, \mathcal{A}, P)$ is called a fuzzy probability space.

We can observe all of these are straightforward generalizations of the analogous properties of ordinary probability measures. Properties of set-formed fuzzy probability by Yager(1979), however, is not nice property in the case of monotonicity and continuity. Since Klement(1982) touches upon the problem, it is fulfilled by modifying Yager's fuzzy probability. He generates fuzzy probability, and deals with the sum and multiplication of a fuzzy set by a real number for nice measure properties.

3. PROBABILITY OF A FUZZY EVENT

3.1 Scalar Fuzzy Probability

L.A. Zadeh(1968) proposed a definition for the probability of fuzzy subsets, specially it is defined in such a manner as to be a number. Probability space will be assumed to be a triplet $(R^n, \mathcal{A}, P)$, where $\mathcal{A}$ is true σ-field of Borel set in R^n and P is a probability measure over R^n. A point in R^n will be denoted by x.

Consider a set $A \in \mathcal{A}$, then the probability of A can be expressed as

$$P(A) = \int_{R^n} \mu_A(x) dP = E(\mu_A)$$

where μ_A denotes the characteristic function of A and $E(\mu_A)$ is the expectation of μ_A. This is equation that can be generalized to fuzzy events through the use of the concept of a fuzzy set. The fuzzy set A is defined by $\mu_A : R^n \rightarrow [0,1]$ which associates with each $x \in R^n$, its "grade of membership", $\mu_A(x)$ in A.

Definition 3.1.1. Fuzzy event in R^n is a fuzzy set A in R^n whose membership function, μ_A, is Borel measurable. Then the probability of a fuzzy event A is defined by the Lebesque-Stieltjes integral

$$P(A) = \int_{R^n} \mu_A(x) dP = E(\mu_A)$$

Example 1 If $X = \{x_1, x_2, x_3, x_4\}$ is a set of people and A is the fuzzy set of tall people such that

$$A = \left\{ \frac{0.7}{x_1}, \frac{0.3}{x_2}, \frac{0.5}{x_3}, \frac{1}{x_4} \right\}$$

where x_i is an element of a universe of discourse X, a value in $[0,1]$ is the grade of membership in A and ',' denotes the union and '–' denotes the separating symbol, not division operator. If we perform an experiment in which we randomly select a element $x_i \in X$, where $P(x_1) = 0.5, P(x_2) = 0.3, P(x_3) = 0.1$ and $P(x_4) = 0.1$, then the probability of the fuzzy event, selection of a tall person would be

$$P(A) = \int \mu_A(x) dP = (0.7)(0.5) + (0.3)(0.3) + (0.5)(0.1) + (1)(0.1) = 0.59$$

This definition means that if x_i happens, then $\mu_A(x_i)$ is the degree of tall people that occur. According to this, expectation of these degrees is the probability of A.

There are several basic notions related to fuzzy sets. First, we shall discuss monotonicity between probability of two fuzzy subsets. From the notion of the containment to two fuzzy subsets, A, B,

$$A \subseteq B \Rightarrow \mu_A(x) \leq \mu_B(x), \quad x \in X$$

Monotonicity of scalar fuzzy probability of each fuzzy subsets, we have

$$A \subseteq B \Rightarrow P(A) \leq P(B)$$

Also, from the properties of the basic operations between two fuzzy sets, We summarize below.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad ------- \quad (1)$$

$$P(A \oplus B) = P(A) + P(B) - P(A \cdot B) \quad ------- \quad (2)$$

where $\cdot$ denotes product operation. In a similar fashion, (1) and (2) yeild the generalized Boole inequalities for fuzzy sets as following.

$$P(\bigcup_{x=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$$

$$P(A_1 \oplus \cdots) \leq \sum_{i=1}^{\infty} P(A_i)$$

In turn, we will observe that A and B are independent. If $P(A \cdot B) = P(A)P(B)$, then A and B will be said to be independent. Note that in defining independence we employ the product $A \cdot B$ rather than the $A \cap B$. Since conditional probability of A given B is defined by $P(A|B) = P(A \cdot B)/P(B), (P(B) > 0)$, then $P(A|B) = P(A)$, as in the case of independent events. This is the same as result of nonfuzzy independent sets.

3.2 Set-formed Fuzzy Probability in Yager's Definition

One of views has used for handling a fuzzy event is Yager's method based on the extension principle and α-level cut by Zadeh(1975). In proposition "X is A", A will be considered to be a fuzzy event over the unit interval, then probability of A, $P(A)$ is interpreted as linguistic probability such " is likely that " or " is probable ", which is a fuzzy subset over unit interval.

If $\alpha \in [0, 1]$, $\alpha \cdot A$ is a fuzzy subset of X whose membership function is $\alpha \cdot \mu_A(x)$. From the α-level set of A, A_α is the ordinary subset of X defined by

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}$$

Any fuzzy subset A can be expressed by using A_α.

$$A = \bigcup_{\alpha=0}^{1} \alpha \cdot A_\alpha$$

If X is a set on which we have a probability measure, then probability of A_α is $P(A_\alpha)$.

Assume Y and W are two sets. Let $A \subset Y, G \subset Y$ such as A is a fuzzy subset and G is crisp set. If f is a mapping, $f : G \rightarrow W$, we can extend f to fuzzy subset of Y as follows

$$f(A) = \bigcup_{\alpha=0}^{1} \alpha \cdot f(A_\alpha)$$

In particular, if A is a fuzzy event of X, we can define $P_Y(A)$ by replacing mapping f to probability function P which associates with Borel sets of X as

$$P_Y(A) = \bigcup_{\alpha=0}^{1} \alpha \cdot P(A_\alpha)$$

This can be expressed by a fuzzy subset of $[0,1]$ as

$$\{P(A_\alpha)\} = \left\{\frac{1}{P(A_\alpha)}\right\}$$

Then

$$P_Y(A) = \bigcup_{\alpha=0}^{1} \alpha \left\{\frac{1}{P(A_\alpha)}\right\}$$

where $P(A_\alpha)$ is a element of $[0,1]$, which is summed probabilities of each elements belong to A_α. Then, since $P(A_\alpha) \in [0,1], P_Y(A)$ should be a fuzzy number of $[0,1]$.

Example 2 Recall the previous Example 1.

First, yield α-level sets as follows

$$\begin{array}{lllll} A_{\alpha_1} & = & A_{0.3} & = & \{x_1, x_2, x_3, x_4\}, & \alpha \leq 0.3 \\ A_{\alpha_2} & = & A_{0.5} & = & \{x_1, x_3, x_4\}, & 0.3 < \alpha \leq 0.5 \\ A_{\alpha_3} & = & A_{0.7} & = & \{x_1, x_4\}, & 0.5 < \alpha \leq 0.7 \\ A_{\alpha_4} & = & A_1 & = & \{x_4\}, & 0.7 < \alpha \leq 1 \end{array}$$

We also note that

$$P(A_{\alpha_i}) = \sum P(x_i), \ x_i \in A_\alpha, \ i = 1, \cdots, 4$$

Therefore,

$$P_Y(A) = \bigcup_{\alpha=0}^{1} \alpha \left\{ \frac{1}{P(A_\alpha)} \right\} = \left\{ \frac{0.3}{1}, \frac{0.5}{0.7}, \frac{0.7}{0.6}, \frac{1}{0.1} \right\}$$

We shall investigate some properties of the fuzzy probability $P_Y(A)$. A more detailed discussion of these and other notions may be found in Yager(1979).

Theorem 3.2.1. Assume A and B are two fuzzy subsets of X, universe of discourse. If $B \subset A$, then $P_Y(A) \not\subset P_Y(B)$.

Theorem 3.2.2. If $q \in [0,1]$ is in the support of $P_Y(A)$ and $P_Y(B)$ and $B \subset A$, then

$$\mu_{P(B)}(q) \leq \mu_{P(A)}(q)$$

If $P_Y(A)$ and $P_Y(B)$ are not necessarily the same supports, there is no need for $P_Y(B) \subset P_Y(A)$, even though $B \subset A$. An example will help clarify this procedure.

Let

$$A = \left\{ \frac{1}{x_1}, \frac{0.6}{x_2}, \frac{0.4}{x_3}, \frac{0.3}{x_4} \right\}$$
$$B = \left\{ \frac{0.2}{x_1}, \frac{0.5}{x_2}, \frac{0.3}{x_3}, \frac{0.1}{x_4} \right\}$$

with $P(x_1) = 0.2, P(x_2) = 0.3, P(x_3) = 0.4, P(x_4) = 0.1$

Then

$$P_Y(A) = \left\{ \frac{0.3}{1}, \frac{0.4}{0.9}, \frac{0.6}{0.5}, \frac{1}{0.2} \right\}$$
$$P_Y(B) = \left\{ \frac{0.1}{1}, \frac{0.2}{0.9}, \frac{0.3}{0.7}, \frac{0.5}{0.3} \right\}$$

Therefore, $P_Y(B) \not\subset P_Y(A)$, since $P_Y(A)$ and $P_Y(B)$ have different supports. If $P_Y(A)$ and $P_Y(B)$ have same supports, $P_Y(B) \subset P_Y(A)$ by Theorem 3.2.2. Consequently, we have assertion that if $B \subset A$, then $P_Y(A) \not\subset P_Y(B)$. The following theorem proposes condition to add monotonicity in Yager's methodology.

Theorem 3.2.3. $P_Y(A)$ and $P_Y(B)$ have the same support if $\mu_A(x)$ and $\mu_B(x)$ are related by a strictly monotonic order preserving transfomation as following

$$\mu_A(x_i) > \mu_A(x_j) \text{ iff } \mu_B(x_i) > \mu_B(x_j) \text{ and}$$

$$\mu_A(x_i) = \mu_A(x_j) \text{ iff } \mu_B(x_i) = \mu_B(x_j)$$

Under the conditions of Theorem 3.2.2 and Theorem 3.2.3, we can observe that fuzzy probability has monotonicity. But, even if above condtions, Yager's method does not hold continuity. There are other properties as following.

Theorem 3.2.4. Let A and B be two fuzzy events of X, with probabilities $P(A)$ and $P(B)$, respectively. Then

(i) $P_Y(A \cup B) = \bigcup_{\alpha=0}^{1} \left\{ \frac{\alpha}{P(A_\alpha) + P(B_\alpha) - P(A_\alpha \cap B_\alpha)} \right\}$

(ii) $P_Y(A \cap B) = \bigcup_{\alpha=0}^{1} \left\{ \frac{\alpha}{P(A_\alpha \cap B_\alpha)} \right\}$

(iii) if A and B are independent, then

$$P_Y(A \cap B) = \bigcup_{\alpha=0}^{1} \left\{ \frac{\alpha}{P(A_\alpha) \cdot P(B_\alpha)} \right\}$$

3.3 Modified Fuzzy Probability

Klement(1982) suggests a modification of Yager's fuzzy probability definition which leads to a piecewise continuous fuzzy subset defined over the whole unit interval. It provides an alternative definition for a fuzzy probability of a fuzzy event. We shall denote this fuzzy probability as $P^*(A)$. $P^*(A)$ can be defined from $P_Y(A)$ as follows by Klement(1982).

Definition 3.3.1. Let $w_1 < w_2 < w_3 < \cdots < w_n$ be the support of $P_Y(A)$ and let $w_o = 0$ and $w_{n+1} = u > 1$. For all w, $P^*(A)$ is defined as

$$P^*(A)(w) = \sup_{\alpha}\{\alpha | P(A_\alpha) \geq w\}, \quad w \in [0, 1].$$

We can interpret $P^*(A)(w)$ as the truth of the proposition "the probability of A is at least w".

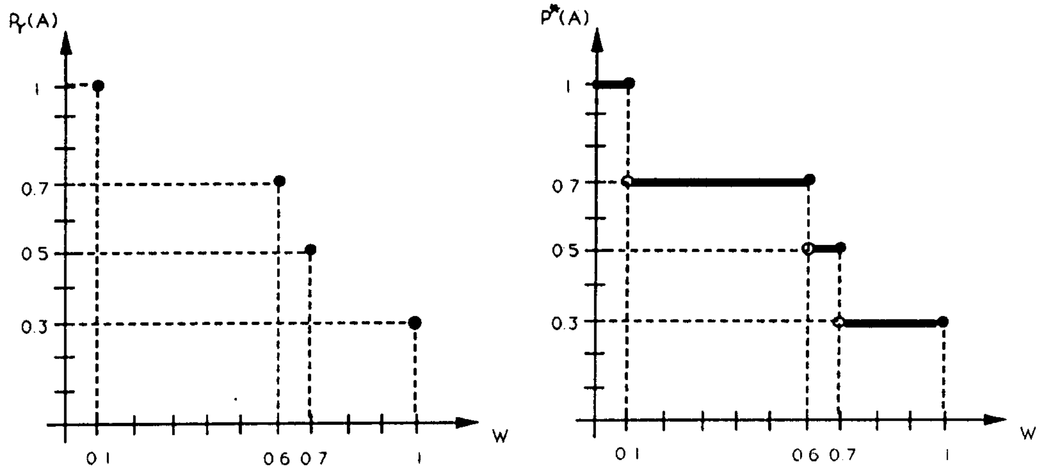
Theorem 3.3.1. $P^*(A)$ and $P_Y(A)$ coincide at the single points which belong to I. If A is a fuzzy event and $w \in I$ such that $w = P(A_\alpha)$, then we get

$$P^*(A)(w) = P_Y(A)(w)$$

Example 3 Recall Example 2. There exists a number such that $w = P(A_\alpha)$, $\forall w \in I$. Thus, for such w, we get following results.

$$P^*(A)(w) = \begin{cases} 1, & w \in [0, 0.1] \\ 0.7, & w \in (0.1, 0.6] \\ 0.5, & w \in (0.6, 0.7] \\ 0.3, & w \in (0.7, 1] \end{cases} \quad P_Y(A)(w) = \begin{cases} 1, & w = 0.1 \\ 0.7, & w = 0.6 \\ 0.5, & w = 0.7 \\ 0.3, & w = 1 \end{cases}$$

Fig. 1 presents $P^*(A)$ and $P_Y(A)$ are same at w.



< Fig. 1> Fuzzy Probabilities, $P^*(A)$ and $P_Y(A)$

From Definition 3.3.1 and Theorem 3.3.1 , we can observe that $P^*(A)$ and Yager's $P_Y(A)$ have containment relationship

$$P^*(A) \supset P_Y(A)$$

Using this interpretation, we can drive another notion of "the probability of A is at most w".

Consider the complement of A, A' such that $\mu_{A'}(x) = 1 - \mu_A(x)$, $\forall x \in X$. Zimmermann(1988) defined $P_*(A)$ as

$$P_*(A) = 1 - P_*(A')$$

In this situation, we can combine this two probabilities, which denotes $\hat{P}(A)$. $\hat{P}(A)(w)$ means "the probability of A is exactly w". We adopt the following definition by Yager(1984)

$$\hat{P}(A)(w) = min\{P^*(A)(w), P_*(A)(w)\},\ w \in [0,1]$$

Through the follwing example, we can observe the differences of $P^*(A), P_*(A)$ and $\hat{P}(A)$.

Example 4 Let $X = \{x_1, x_2, x_3, x_4\}$ and $A = \left\{\frac{1}{x_1}, \frac{0.7}{x_2}, \frac{0.6}{x_3}, \frac{0.2}{x_4}\right\}$ with $P(x_1) = 0.1, P(x_2) = 0.4, P(x_3) = 0.3, P(x_4) = 0.2$.

For $P^*(A)$,

$$\begin{array}{llllllll} \text{if} & \alpha = [0,0.2], & \text{then} & A_{0.2} = \{x_1,x_2,x_3,x_4\} & \text{and} & P(A_{0.2}) & = & 1 \\ & \alpha = (0.2,0.6], & & A_{0.6} = \{x_1,x_2,x_3\} & & P(A_{0.6}) & = & 0.8 \\ & \alpha = (0.6,0.7], & & A_{0.7} = \{x_1,x_2\} & & P(A_{0.7}) & = & 0.5 \\ & \alpha = (0.7,1], & & A_1 = \{x_1\} & & P(A_1) & = & 0.1 \end{array}$$

Then $P^*(A)$ results from its definition

$$\begin{array}{lllll} \text{for} & w \in (0.8,1], & P^*(A)(w) & = & 0.2 \\ & w \in (0.5,0.8], & P^*(A)(w) & = & 0.6 \\ & w \in (0.1,0.5], & P^*(A)(w) & = & 0.7 \\ & w \in [0,0.1], & P^*(A)(w) & = & 1 \end{array}$$

Since $A' = \left\{\frac{0}{x_1}, \frac{0.3}{x_2}, \frac{0.4}{x_3}, \frac{0.8}{x_4}\right\}$ and $P_*(A) = 1 - P^*(A')$

$$\begin{array}{lllllll} \text{if} & \alpha = 0, & \text{then} & A'_0 = \{x_1,x_2,x_3,x_4\} & P(A'_0) & = & 1 \\ & \alpha = (0,0.3], & & A'_{0.3} = \{x_2,x_3,x_4\} & P(A'_{0.3}) & = & 0.9 \\ & \alpha = (0.3,0.4], & & A'_{0.4} = \{x_3,x_4\} & P(A'_{0.4}) & = & 0.5 \\ & \alpha = (0.4,0.8], & & A'_{0.8} = \{x_4\} & P(A'_{0.8}) & = & 0.2 \\ & \alpha = (0.8,1], & & A'_1 = \emptyset & P(A'_1) & = & 0 \end{array}$$

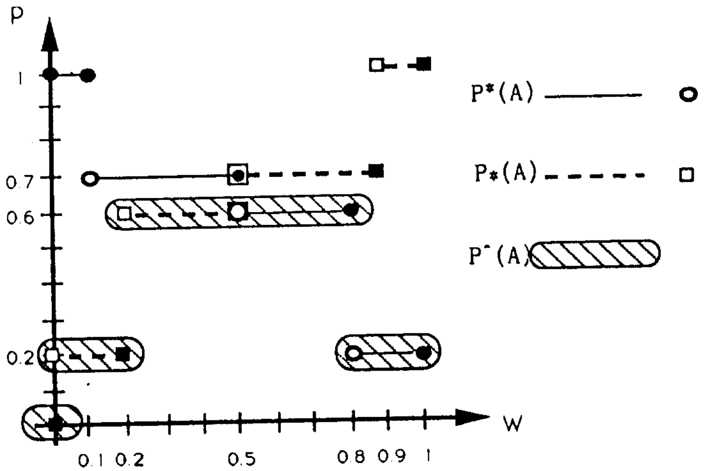
Similary, $P_*(A)$ results from its definition

$$\begin{array}{lllll} \text{for} & w \in (0.9,1], & P_*(A)(w) & = & 1 \\ \text{for} & w \in (0.5,0.9], & P_*(A)(w) & = & 0.7 \\ \text{for} & w \in (0.2,0.5], & P_*(A)(w) & = & 0.6 \\ \text{for} & w \in (0,0.2], & P_*(A)(w) & = & 0.2 \\ \text{for} & w = 0, & P_*(A)(w) & = & 0 \end{array}$$

Since $\hat{P}(A) = P^*(A) \cap P_*(A)$, we get

$$\hat{P}(A)(w) = \begin{cases} 0, & w = 0 \\ 0.2, & w \in (0,0.2] \\ 0.6, & w \in (0.2,0.8] \\ 0.2, & w \in (0.8,1] \end{cases}$$

Fig.2 presents differences of $P^*(A), P_*(A)$ and $\hat{P}(A)$.



< Fig. 2 > Fuzzy Probabilities, $P^*(A), P_*(A)$, and $\hat{P}(A)$

Lemma 3.3.1. For any subset A of X, $P^*(A)$ is a monotonically decreasing function of $[0,1]$.

(proof) Since $P^*(A)(w) = \sup_\alpha\{\alpha | P(A_\alpha) \geq w\}$, the result follows from the monotonity of probability and the fact that for $\alpha_2 < \alpha_1$, $A_{\alpha_2} \supset A_{\alpha_1}$.

Theorem 3.3.2. If $A \subset B$ for, two fuzzy sets A, B, then for $w \in I, P^*(A)(w) \leq$

$P^*(B)(w)$ and thus $P^*(A) \subset P^*(B)$.

(proof) For $A \subset B$, $A_\alpha \subset B_\alpha$ and hence $P(A_\alpha) \leq P(B_\alpha)$. Since $P^*(A)(w) = \sup_\alpha\{\alpha | P(A_\alpha) \geq w\}$ and similary, $P^*(B)(w) = \sup_\alpha\{\alpha | P(B_\alpha) \geq w\}$.

Therefore for some interval $w \in [0,1]$, from the monotonicity of α -level sets, $P^*(A)(w) \leq P^*(B)(w)$.

3.4 Fuzzy Probability using FG-count

Another concept of fuzzy probability by Zadeh(1984) was suggested with "count" of elements of a fuzzy set. To refer to this count we can write sigma-count of A

$$\sum \text{Count}(A) \equiv \sum \mu_A(x_i), \text{ where } \sum \text{ denotes arithmetic-sum.}$$

Now, let Count(A_α) denote the count of elements of the nonfuzzy set A_α. Then the FG-count of A, where F stands for fuzzy and G stands for greater than, is defined as the fuzzy number

$$FG\text{-count}(A) = \frac{\sum \alpha}{\text{Count}(A_\alpha)}$$

We assume that any gap in the Count(A_α) may be filled by a lower count with same α.

Definition 3.4.1. Let x be finite set with cardinality m and A be $A \subset X$. Fuzzy probability of A by using FG-count, $FP, FP(A) = FG\text{-count}(A)/m$. Then, for all $\alpha \in [0,1]$,

$$FP(A) = \bigcup_\alpha \left\{ \frac{\alpha}{\text{Count}(A_\alpha)/m} \right\}$$

To illustrate the use of concepts defined above, we shall show as following example.

Example 5 If an urn contains m balls of various sizes of which several large, "What is the probability that a ball drawn at random is large?"

First, yield FG-count(LARGE) by fuzzy subset LARGE such as

$$\text{LARGE} = \left\{ \frac{0.8}{b_1}, \frac{0.5}{b_2}, \frac{0.6}{b_3}, \frac{1}{b_4}, \frac{1}{b_5}, \frac{0}{b_6}, \frac{1}{b_7}, \frac{1}{b_8}, \frac{1}{b_9}, \frac{0.1}{b_{10}} \right\}$$

$$m = 10$$

Then

$$FG\text{-count(LARGE)} = \left\{\tfrac{1}{5}, \tfrac{0.8}{6}, \tfrac{0.6}{7}, \tfrac{0.5}{8}, \tfrac{0.1}{9}\right\}$$

$$= \left\{\frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{0.8}{6}, \frac{0.6}{7}, \frac{0.5}{8}, \frac{0.1}{9}, \frac{0}{10}\right\}$$

Therefore

$$FP\text{-(LARGE)} = FG - \text{count(LARGE)}/10$$

$$= \left\{\frac{1}{0}, \frac{1}{0.1}, \frac{1}{0.2}, \frac{1}{0.3}, \frac{1}{0.4}, \frac{1}{0.5}, \frac{0.8}{0.6}, \frac{0.6}{0.7}, \frac{0.5}{0.8}, \frac{0.1}{0.9}, \frac{0}{1}\right\}$$

But given condition "SEVERAL" restricts FP(LARGE) to compatibility(Yager (1979), Zadeh(1978)) of FP(LARGE) and $\mu_{\text{SEVERAL}}(x)$. The compatibility of fuzzy subsets A with B, $A \subset X$ and $B \subset X, \forall\, x \in X$, is $A(x)/B(x)$.
Let SEVERAL be a fuzzy number such as

$$\text{SEVERAL} = \left\{\frac{0.4}{3}, \frac{0.8}{4}, \frac{1}{5}, \frac{1}{6}, \frac{0.6}{7}, \frac{0.3}{8}\right\}$$

To get same base values both $FP(A)$ and $\mu_{\text{SEVERAL}}(x)$, we take $\mu_{SEVERAL}(x)/m$,

$$FP(A)/(\mu_{\text{SEVERAL}}(x)/m) = \left\{\frac{1}{0.4}, \frac{1}{0.8}, \frac{1}{1}, \frac{0.6}{0.6}, \frac{0.5}{0.3}\right\}$$

This is different fuzzy probability by Zadeh's definition in early, which is a fuzzy number not single real number.

Furthermore, conditional fuzzy probability A given B is defined as following.

$$FP(A|B) = \bigcup_{\alpha} \left\{ \frac{\alpha}{\text{Count}(A_\alpha \cap B_\alpha)/\text{Count}(B_\alpha)} \right\}$$

where joint fuzzy probability of A and B is $FP\{A \cap B\} = FG$ -count$(A \cap B)/m$. Also, the fuzzy events A, B will be said independent if $FP\{A \cap B\} = FP\{A\} \cdot FP\{B\}$.

Properties of typical probability are holded in the notion of FP with respect to fuzzy events. It implies that FP is a measure which extends typical probability theory to fuzzy theory through FG-count.

4. CONCLUSION

We have introduced various methologies for obtaing fuzzy measure about the probability of fuzzy subsets in the uncertainty. Zadeh suggested two views which one is scalar fuzzy probability, the other is set-formed fuzzy probability. Early, Zadeh's probability definition of fuzzy event which leads to a crisp number. After, he suggested a fuzzy number defined over fuzzy probability of fuzzy event. Zadeh's later work on the related fuzzy probability is based on the cardinality of a fuzzy subset. The approach suggested by Yager is based upon the extension principle. His method is different Zadeh's definition, which leads to a conjunctive fuzzy subset whose membership grade at each $w \in [0,1]$. It indicates the truth of the assertion that "The probability of A exactly equals w". Since Yager's method(1979) does not hold monotonicity and continuity which are important conditions of measures, more detail conditions were needed. And so, Klement suggested a modification of Yager's definition which leads to a piecewise continuous fuzzy subset on $[0,1]$. Interpretation for Klement modification is "the probability of A is at least w" and the case of Zimmermann derived from this modification is interpreted as "the probability of A is at most w", $w \in [0,1]$. Consequently, through alternative definitions for the probability of a fuzzy event, we shall conclude that Klement's concept for fuzzy probability is consistent with our intuitive thinking process and has the nice mathematical properties-boundedness, monotonicity and continuity on interval $[0,1]$.

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