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ON ORTHOGONALITY AND BALANCING IN GENERALIZED CYCLIC FACTORIAL EXPERIMENTS

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ABSTRACT

The concept of Balanced Factorial Experiments(BFE) was introduced by Shah(1958). The conditions for BFE were set up by Kurkjian and Zelen(1963) and Kshirsagar(1966). Generalized Cyclic Factorial Experiment(GCFE), which is more wide class of designs than BFE, do not satisfy the condition of BFE. So all contrasts belonging to the same interaction are not estimated with equal variance. The main purpose of this paper is to show that GCFE have orthogonal factorial structure and the scheme of the size of variances for all normalized contrasts in GCFE is similar to the original intra-block association scheme.

1. INTRODUCTION

When a factorial experiment is arranged as an incomplete block design, it is freqently desirable to use a design which admits an orthogonal analysis of the main and interaction effects. A set of sufficient conditions for the designs to have orthogonal factorial structure was given by Cotter, John and Smith(1973). But there exist many designs which are orthogonal yet not satisfing above sufficient conditions.

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Mukerjee(1979) introduced a set of necessary and sufficient conditions for orthogonal factorial structure which are useful for construction class of such designs.

The concept of factorial balancing was introduced by Shah(1958). Kurkjian and Zelen (1963) defined a class of designs having Property A and proved the Property A designs are factorially balanced. Kshirsagar (1966) proved the designs with factorial balanced necessarily have Property A. Cyclic Factorial Association Scheme (CFAS) for Generalized Cyclic Factorial Experiments (GCFE) was introduced by Paik(1985). Here, GCFE is equivalent to "Generalized Cyclic Designs in Factorial Experiments" termed by John(1973).

It is shown that CFAS possesses the structure K defined by Mukerjee(1979), therefore inter-effect orthogonality holds in Generalized Cyclic Factorial Experiments. Also, it is shown that all normalized contrast belonging to the same interaction in Generalized Cyclic Factorial Experiments are estimated with partially balanced variances and this partially balanced scheme is closely related to the original intra-block Cyclic Factorial Association Scheme.

2. ORTHOGONALITY IN GENERALIZED CYCLIC FACTORIAL EXPERIMENTS

2.1 Orthogonal Factorial Structure

In a factorial experiment involving m factors $F_1, F_2, \ldots, F_m$ at $s_1, s_2, \ldots, s_m$ levels ($s_i \geq 2, 1 \leq i \leq m$). The $v = \prod_{i=1}^{m} s_i$ treatments are allocated in b blocks of size $k_1, k_2, \cdots, k_b$, the j-th treatment being repeated r_j times, where $j = (j_1, j_2, \cdots, j_m), j_i$ being the level of F_i in the j-th treatment. $N_{v\times b} = (n_{ij})$ is the incidence matrix of the design, n_{ij} denoting the number of times, the i-th treament occurs in j-th block.

We assume that the yield of a plot in the j-th block having the i-th treatment is $y_{ij} = \mu + \tau_i + b_j + e_{ij}$, where μ is overall effect, τ_i is the effect of the i-th treatment, b_j is the effect the j-th block and e_{ij} is the error term, all e_{ij} are assumed to be independent normal variate with zero mean and variance σ^2. Let T_i be the total yield of all plots receiving the i-th treatment combination and B_j the total yield of all plots in the j-th block. $\mathbf{T}$ and $\mathbf{B}$ are vectors of treatment and block total respectively. And let $\hat{\tau}_i$ be a least square solution of τ_i in the normal equation.

Then the reduced normal equation for the treatment effects in the intra-block model, eliminating block effects are

$$A\hat{\tau} = \mathbf{Q} \tag{2.1}$$

where $A = diag(r_1, r_2, \ldots, r_v) - N\ diag(k_1^{-1}, k_2^{-1}, \ldots, k_b^{-1})N'$.

$$\mathbf{Q} = \mathbf{T} - N\ diag(r_1, r_2, \ldots, r_v)\ \mathbf{B}$$

Here $\mathbf{Q}$, τ, and $\hat{\tau}$ denote the column vectors $(Q_1, Q_2, \ldots, Q_v)'$, $(\tau_1, \tau_2, \ldots, \tau_v)'$ and $(\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_v)'$ respectively. And the variance matrix of $\mathbf{Q}$ is

$$\sigma^2 A \tag{2.2}$$

In analysing the result of a factorial experiments, we are usually interested in drawing conclusions about contrasts which belong to different factorial effects.

In this context, a great simplication occurs if inter-effect orthogonality hold in the sense Best Linear Estimate of estimable contrasts belonging to different effects are mutually orthogonal.

A set of sufficient conditions for a design to have orthogonal factorial structure was given by Cotter, John and Smith(1973). But there exist many designs which were orthogonal yet do not satisfy these conditions. Mukerjee(1979) gave a set of necessary and sufficient conditions for orthogonal factorial structure which are useful for construction of the class of such designs.

Consider a $v \times v$ matrix A, where $v = s_1 \times s_2 \times, \ldots, \times s_m (s_i > 2$ for all $i)$, and it will be said to have structure K, if A is expressible as a linear combination for Kronecker Products of permutation matrices of orders $s_1, s_2, \ldots, s_m$, i.e. if

$$A = \sum_{j=1}^{w} \xi_j \{R_{j1} \otimes R_{j2} \otimes, \ldots, \otimes R_{jm}\} \tag{2.3}$$

where w is some positive integer, $\xi_1, \xi_2, \ldots, \xi_w$ are some numbers and for each j, R_{ji} is some $s_i \times s_j$ permutation matrix, $1 < i < m$.

It is well known that the square matrix with all row sums and column sums equal is called proper matrix. Mukerjee(1979) showed any proper matrix can be expressed as a linear combination of permutation matrices of the same order. So, (2.3) is expressible as a linear combination of Kronecker Product of proper matrix.

Mukerjee(1979) also proved that for a connected equireplicate block design inter-effect orthogonality hold if and only if

$$N diag(k_1^{-1}, k_2^{-2}, \ldots, k_b^{-1})N' \text{ possess structure } K. \tag{2.4}$$

2.2 Generalized Cyclic Factorial Experiments and Structure K

In a factorial experiment with m factors $F_1, F_2, \ldots, F_m$ at $s_1, s_2, \ldots, s_m$ levels respectively. Consider the following association scheme : the two treatment combinations, $(i_1, i_2, \ldots, i_m)$ and $(j_1, j_2, \ldots, j_m)$ are $(p_1, p_2, \ldots, p_m)$-th associates, where $p_k = 0, 1, 2, \ldots, s_{k-1}$ for $k = 1, 2, \ldots, m$, if

$$(i_1, i_2, \ldots, i_m) + (p_1, p_2, \ldots, p_m) = (j_1, j_2, \ldots, j_m) \tag{2.5}$$

where $i_k + p_k = j_k \bmod s_k (k = 1, 2, \ldots, m)$. This association was designated by Paik(1985) as Cyclic Factorial Association Scheme (CFAS). In a block design, with reference to any specified treatment, the remaining $v - 1$ fall into m(say) sets, the $(p_1, p_2, \ldots, p_m)$-th associate of which occurs with the specified treatment in $\lambda(p_1, p_2, \ldots, p_m)$ blocks and the numbers $\lambda(p_1, p_2, \ldots, p_m)$ are the same regardless of the treatments specified as long as they are $(p_1, p_2, \ldots, p_m)$-th association.

By definition, two treatments $(i_1 - p_1, i_2 - p_2, \ldots, i_m - p_m)$ and $(i_1, i_2, \ldots, i_m)$ are $(p_1, p_2, \ldots, p_m)$-th association and also $(s_1 - p_1, s_2 - p_2, \ldots, s_m - p_m)$-th association, because $(i_1, i_2, \ldots, i_m) + (s_1 - p_1, s_2 - p_2, \ldots, s_m - p_m) = (i_1 - p_1, i_2 - p_2, \ldots, i_m - p_m)$.

Therefore, in the PBIB having CFAS, thee following relationship holds:

$$\lambda(s_1 - p_1, s_2 - p_2, \ldots, s_m - p_m) = \lambda(p_1, p_2, \ldots, p_m) \tag{2.6}$$

In a PBIB design having CFAS design, it is well known that the structural matrix NN' can be expressed as following

$$NN' = \sum_{i_1=0}^{s_1-1} \sum_{i_2=0}^{s_2-1} \cdots \sum_{i_m=0}^{s_m-1} \lambda(i_1, i_2, \ldots, i_m) \prod_{j=1}^{m} \otimes R_{n_j}^{i_j} \tag{2.7}$$

where $\lambda(i_1, i_2, \ldots, i_m)$ are constant which correspond to the first row elements of NN' and $R_{n_j}^{i_j}$ is $n_j \times n_j$ circulant matrix whose first row has unity in the i_j-th column and zero elsewhere, where $i = 0, 1, 2, \ldots, m-1$. This structural property was termed Property C by Paik(1985). For GCFE the matrices NN' in (2.7) and A in equation (2.1) are all circulant. It is sufficient to notice that structural type in (2.7) can be expressed by structure K type (2.3). So, we can say Generalized Cyclic Factorial Experiments possesses structure K. In conclusion, Inter-effect orthogonality holds in Generalized Cyclic Factorial Experiments.

3. BALANCING IN GENERALIZED CYCLIC FACTORIAL EXPERIMENTS

Factorial balancing means all the single degree of freedom contrasts belonging to a particular interaction effect are estimated with the equal variance.

This concept was introduced by Shah(1958). Kurkjian and Zelen(1963) defined a class of designs designated as Property A designs and proved the Property A designs are factorially balanced. Kshirsagar(1966) proved that designs with factorial balance necessarily have Property A. Recently Lewis and Tuck(1985) considered factorially balanced paired comparison designs.

Lewis and Tuck(1985) provided a table of 74 generalized cyclic paired comparison designs that are factorially balanced. Gupta(1987) proposed the algorithm for generating paired Comparision Generalized Cyclic Design with factorial balance.

Gupta's algorithm generates generalized cyclic paired comparison designs with factorial balance. But all Cyclic Factorial Experiments are not always factorially balanced. Designs which is generated by Gupta's algorithm do not include all Cyclic Factorial Experiments.

It is well-known that the variance of normalized contrast is determined by eigen-values of matrix A in equation (2.1).
A solution of equation (2.1) is given by

$$\hat{\tau} = \Omega \mathbf{Q} \tag{3.1}$$

where Ω is generalized inverse of matrix A in equation (2.1).

If L is $q \times v$ contrast matrix such that $L = L\Omega A$, a unique estimator of $L\tau$ is given by $L\hat{\tau}$, which variance-covariance matrix, $V(L\hat{\tau}) = L\Omega L'\sigma^2$.

We consider $v \times 1$ coefficient vector ℓ which is non-null and $\ell'\ell = 1$, then the contrast $\ell'\tau$ will be called a normalized contrast.

For a normalized contrast Shah(1958) obtained the following results,

$$\left.\begin{array}{rcl} V(\ell_i'\hat{\tau}) & = & \sigma^2/\theta_i \\ Cov(\ell_i'\hat{\tau}, \ell_j'\hat{\tau}) & = & 0 \end{array}\right] \tag{3.2}$$

where θ_i is i-th eigen-value of A-matrix and ℓ_i is the corresponding eigen-vector.

In equation(3.1), we can take $\Omega^{-1} = A + J$, where J is $v \times v$ matrix all element unity. And the matrix $(A + J)$ also has the Property C.

The $(j_1, j_2, \ldots, j_m)$-th element of the first row of $(A + J)^{-1}$ can be drived as following [Paik (1985)]

$$c(j_1, j_2, \ldots, j_m) = (\prod_{i=1}^{m} s_i)^{-1} \sum_{i_1=0}^{s_1-1} \sum_{i_2=0}^{s_2-1} \cdots \sum_{i_m=0}^{s_m-1} \theta^{-1}(i_1, i_2, \ldots, i_m) \cos(\sum_{k=1}^{m} \frac{2 i_k j_k \pi}{s_k}) \quad (3.3)$$

where

$$\theta(i_1, i_2, \ldots, i_m) = \sum_{j_1=0}^{s_1-1} \sum_{j_2=0}^{s_2-1} \cdots \sum_{j_m=0}^{s_m-1} c^*(j_1, j_2, \ldots, j_m) \cos(\sum_{k=1}^{m} \frac{2 i_k j_k \pi}{s_k}) \quad (3.4)$$

where $c^*(j_1, j_2, \ldots, j_m)$ is the $(j_1, j_2, \ldots, j_m)$-th element of the first row of $(A + J)$.

In the case of m multi-nested symmetric circulant matrix, the eigen-value is given by (3.4) and

$$\begin{aligned} \theta(s_1 - j_1, s_2 - j_2, \ldots, s_m - j_m) &= \sum_{i_1=0}^{s_1-1} \sum_{i_2=0}^{s_2-1} \cdots \sum_{i_m=0}^{s_m-1} c^*(i_1, i_2, \ldots, i_m) \cos[\sum_{k=1}^{m} \frac{2(s_k - j_k) i_k \pi}{s_k}] \\ &= \sum_{i_1=0}^{s_1-1} \sum_{i_2=0}^{s_2-1} \cdots \sum_{i_m=0}^{s_m-1} c^*(i_1, i_2, \ldots, i_m) \cos[\sum_{k=1}^{m} \frac{2 j_k i_k \pi}{s_k}] \\ &= \theta(j_1, j_2, \ldots, j_m) \quad (3.5) \end{aligned}$$

Similarly, from (3.3), we obtain the following

$$c(s_1 - j_1, s_2 - j_2, \ldots, s_m - j_m) = c(j_1, j_2, \ldots, j_m) \quad (3.6)$$

Using relationships (2.6), (3.5), and (3.6), we now state the following theorem.

Theorem 3.1. All normalized contrasts belonging to the same interaction in Generalized Cyclic Factorial Experiments are estimated with partially balanced variances and this partially balanced scheme is similar to the original intra-block Cyclic Factorial Association sheme.

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