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ASYMPTOTIC DISTRIBUTION OF LIKELIHOOD RATIO STATISTIC FOR TESTING MULTISAMPLE SPHERICITY

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ABSTRACT

In this paper, asymptotic expansions of the distribution of the likelihood ratio statistic for testing multisample sphericity have been derived in the null and nonnull cases when the alternatives are close to the null hypothesis. These expansions are obtained in the form of series of beta distributions.

1. INTRODUCTION

Let $X_{ij}, j = 1, \ldots, N_i$ be a random sample from $N_p(\mu_i, \Sigma_i)$ where μ_i and Σ_i are unknown, $i = 1, \ldots, q$. Also, let

$$A_i = \sum_{j=1}^{N_i}(X_{ij} - X_{i\cdot})(X_{ij} - X_{i\cdot})' \text{ where } X_{i\cdot} = N_i^{-1}\sum_{j=1}^{N_i} X_{ij}.$$

Then $A_i \sim W_p(n_i, \Sigma_i)$, where $n_i = N_i - 1, i = 1, \ldots, q$. Let H denote the hypothesis of multisample sphericity, i.e.

$$H : \Sigma_1 = \cdots = \Sigma_q = \sigma^2 I_p$$

where $\sigma^2 > 0$ is an unknown scalar and I_p is the identity matrix of order p. Such an hypothesis arises in repeated measures design with two or more repeated factors,

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where if the assumption of homogeneity of group covariance matrices can not be made a priori, it needs to be tested.

Pillai and Young(1973) considered the statistic $R_2 = \max_{1\le i\le q} T_i / \min_{1\le j\le q} T_j$ where $T_i = trA_i/n_i$ and derived certain distributional results by noting that R_2 is the same as Hartley's F_{max} statistic. Mendoza(1980) derived the modified likelihood ratio statistic Λ^*, its null moments and gave a one term approximation to its null distribution using Box's method. Gupta and Nagar(1987a) derived the h^{th} nonull moment of Λ^* in series involving zonal polynomials and derived its exact null distributions in series involving psi-, generalized Riemann zeta-, and H-functions. Gupta and Nagar(1988) further derived asymptotic nonnull distribution of a multiple of $-2\ln\Lambda^*$, for certain alternative, in series involving chi-square distribution (also see Gupta and Nagar(1987b)).

When $q = 1$, the hypothesis of multisample sphericity reduces to the Mauchly's sphericity hypothesis $H : \Sigma_1 = \sigma^2 I_p$ and the statistic Λ^* in this case will be the sphericity test criterion. Various distributional results are available in this case (see Sugiura(1969), Nagao(1970), Girshick(1941), Muirhead(1976), Nagarsenker and Pillai(1973), Gupta(1977), Khatri and Srivastava(1971, 1974), Kulp and Nagarsenker (1983)). Kulp and Nagarsenker(1983) derived asymptotic expansions of the null and nonnull (under certain alternatives) distributions of $U = (\Lambda^{*2/n_1})^{1/s}$, where $s(>0)$ is an adjustable constant, in series involving beta distributions. They also computed certain significance points using one term approximation of their results for $p = 2, 3,$ and 4 and compared them with the results obtained by using the first term of Sugiura's(1969) expansion (chi-square approximation) and the exact results given by Nagarsenker and Pillai(1973). They showed that the beta series approximation is preferred. They also computed power of the test for $p = 2$ using their beta series approximation and Khatri and Srivastava's(1974) chi-square approximation.

In this paper we give the asymptotic expansions of the null and nonnull distributions of $U = (\Lambda^{*2/n_0})^{1/s}, n_0 = \sum_{i=1}^{q} n_i$, where $s(>0)$ is an adjustable constant, in terms of beta distributions. The expansion in terms of beta distribution should be better than the ones available so far.

First in Section 2, some preliminary results are given. In Section 3, the exact as well as asymptotic distributions of U are derived in a series of beta distributions. Then in Section 4, asymptotic nonnull distribution of the same statistic under certain alternatives has been derived.

2. PRELIMINARIES

Let $C_\kappa(V)$ denote the zonal polynomial, a symmetric function in the roots of the symmetric matrix V, of degree k corresponding to the partition

$$\kappa = (k_1, \ldots, k_p),\ k_1 \geq \cdots \geq k_p \geq 0,\ k_1 + \cdots + k_p = k,\ (a)_\kappa = \prod_{j=1}^{p} \left(a - \frac{j-1}{2}\right)_{k_j}$$

$$(a)_m = a(a+1)\cdots(a+m-1),\ (a)_0 = 1,\ \Gamma_p(a) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma\left(a - \frac{j-1}{2}\right)$$

and $\Gamma_p(a, \kappa) = (a)_\kappa \Gamma_p(a)$. We need the following results in the sequel.

Lemma 2.1. Let $a_1(\kappa) = \sum_{i=1}^{p} k_i(k_i - i)$ and $a_2(\kappa) = \sum_{i=1}^{p} k_i(4k_i^2 - 6ik_i + 3i^2)$, then for any positive definite matrix V, we have

$$(\ i)\ \sum_\kappa a_1(\kappa) C_\kappa(V) = k(k-1)(trV^2)(trV)^{k-2}$$

$$(\ ii)\ \sum_\kappa a_1^2(\kappa) C_\kappa(V) = k(k-1)[\{trV^2 + (trV)^2\}(trV)^{k-2} + 4(k-2)(trV^3)(trV)^{k-3} + (k-2)(k-3)(trV^2)^2(trV)^{k-4}]$$

$$(iii)\ \sum_\kappa a_2(\kappa) C_\kappa(V) = k[(trV)^k + 3(k-1)\{trV^2 + (trV)^2\}(trV)^{k-2} + 4(k-1)(k-2)(trV^3)(trV)^{k-3}]$$

where $C_\kappa(\cdot)$ is the zonal polynomial (James(1964)).

Proof. See Sugiura(1971).

Lemma 2.2. The h^{th} nonnull moment of Λ^* is given by

$$E(\Lambda^{*h}) = \{(n_0p)^{n_0p/2} / \prod_{i=1}^{q} n_i^{n_ip/2}\}^h \prod_{i=1}^{q} \left\{ \frac{|\eta\Sigma_i^{-1}|^{\frac{1}{2}n_i} \Gamma_p[\frac{1}{2}n_i(1+h)]}{\Gamma_p[\frac{1}{2}n_i]} \right\} \cdot \sum_{k=1}^{\infty} \sum_\kappa \sum_{\kappa^{(1)}} \cdots \sum_{\kappa^{(q)}} \frac{(\frac{1}{2}n_1(1+h))_{\kappa^{(1)}}}{k^{(1)}!} \cdots \frac{(\frac{1}{2}n_q(1+h))_{\kappa^{(q)}}}{k^{(q)}!} \cdot C_{\kappa^{(1)}}(I_p - \eta\Sigma_1^{-1}) \cdots C_{\kappa^{(q)}}(I_p - \eta\Sigma_q^{-1}) \cdot \frac{\Gamma[\frac{1}{2}n_0p + k]}{\Gamma[\frac{1}{2}n_0p(1+h) + k]} \quad (2.1)$$

where $\kappa = \{k^{(1)}, \ldots, k^{(q)}\}$ such that $k^{(1)} + \cdots + k^{(q)} = k$, $C_{\kappa^{(i)}}(\cdot)$ is the zonal polynomial and η is a positive constant such that $||I_p - \eta\Sigma_i^{-1}|| < 1$.

Proof. See Gupta and Nagar(1987a).

Substituting $\Sigma_1 = \cdots = \Sigma_q = \eta I_p$ in the expression (2.1), we get the h^{th} null moment of Λ^* as follows:

$$E(\Lambda^{*h}|H) = \{(n_0 p)^{n_0 p/2} / \prod_{i=1}^{q} n_i^{n_i p/2}\}^h \prod_{i=1}^{q}\prod_{j=1}^{p}\left\{\frac{\Gamma[\frac{1}{2}n_i(1+h)-\frac{1}{2}(j-1)]}{\Gamma[\frac{1}{2}(n_i-j+1)]}\right\} \cdot \frac{\Gamma[\frac{1}{2}n_0 p]}{\Gamma[\frac{1}{2}n_0 p(1+h)]} \tag{2.2}$$

3. NULL DISTRIBUTION

Let $U = (\Lambda^{*2/n_0})^{1/s}$. Substituting $\left(\frac{2}{n_0}h\right)\frac{1}{s}$ for $h, \gamma_i = \frac{n_i}{n_o}$, $i = 1, \ldots, q$ and then $n_o = m + 2\delta$ in (2.2) and taking inverse Mellin transform of the resulting expression we get the density of U as

$$f(u) = K^*(2\pi\imath)^{-1}\int_{-\imath\infty}^{+\imath\infty}\{\prod_{i=1}^{q}\prod_{j=1}^{p}\Gamma[\gamma_i(\frac{1}{2}m+\frac{h}{s})+\gamma_i\delta-\frac{1}{2}(j-1)]/\Gamma[\frac{1}{2}mp+\delta p+\frac{ph}{s}]\} \cdot \frac{p^{\frac{ph}{s}}}{\prod_{i=1}^{q}\gamma_i^{\frac{\gamma_i ph}{s}}}u^{-h-1}dh, \quad 0 < u < 1 \tag{3.1}$$

where $\imath = (-1)^{\frac{1}{2}}$ and

$$K^* = \Gamma[\frac{1}{2}mp+\delta p]/\prod_{i=1}^{q}\prod_{j=1}^{p}\Gamma[\gamma_i(\frac{1}{2}m+\delta)-\frac{1}{2}(j-1)]. \tag{3.2}$$

Now substituting $\frac{1}{2}m + \frac{h}{s} = \frac{t}{s}$, we have

$$f(u) = K^*((2\pi\imath)^{-1}p^{-\frac{1}{2}mp}\prod_{i=1}^{q}\gamma_i^{\frac{1}{2}mp\gamma_i}u^{\frac{1}{2}ms-1}\int_{c-\imath\infty}^{c+\imath\infty}\phi(t)u^{-t}dt, \quad 0 < u < 1, \tag{3.3}$$

where $c = \frac{1}{2}ms$ and

$$\phi(t) = \{p^{\frac{pt}{s}}/\prod_{i=1}^{q}\gamma_i^{\frac{\gamma_i pt}{s}}\} \cdot \{\prod_{i=1}^{q}\prod_{j=1}^{p}\Gamma[\frac{\gamma_i t}{s}+\gamma_i\delta-\frac{1}{2}(j-1)] \;/\; \Gamma[\frac{pt}{s}+\delta p]\} \tag{3.4}$$

Expanding the logarithm of $\phi(t)$ using Barnes' expansion and converting back, one obtains

$$\phi(t) = (2\pi)^{\frac{1}{2}(pq-1)}\prod_{i=1}^{q}\gamma_i^{\gamma_i\delta p-\frac{p(p+1)}{4}}(\frac{t}{s})^{-v}p^{(-\delta p+\frac{1}{2})}[1+\sum_{\alpha=1}^{\infty}\frac{Q_\alpha}{t^\alpha}] \tag{3.5}$$

where $v = \frac{1}{4}qp(p+1) - \frac{1}{2}$,

$$Q_\alpha = \frac{1}{\alpha}\sum_{r=1}^{\alpha} rA_rQ_{\alpha-r}, \quad \alpha = 1,2,\dots \quad Q_0 = 1 \tag{3.6}$$

$$A_r = \frac{(-1)^{r+1}s^r}{r(r+1)}\{\sum_{i=1}^{q}\sum_{j=1}^{p}\gamma_i^{-r}B_{r+1}(\gamma_i\delta - \frac{j-1}{2}) - p^{-r}B_{r+1}(p\delta)\} \tag{3.7}$$

and $B_r(\cdot)$ is the Bernoulli polynomial of degree r and order one.

Substituting (3.5) in (3.3) we get

$$f(u) = K^*(2\pi\imath)^{-1}u^{\frac{1}{2}ms-1}p^{-(\frac{1}{2}mp+\delta p-\frac{1}{2})}\prod_{i=1}^{q}\gamma_i^{\gamma_i p(\frac{1}{2}m+\delta)-\frac{p(p+1)}{4}}s^v$$

$$\cdot(2\pi)^{\frac{1}{2}(pq-1)}\int_{c-\imath\infty}^{c+\imath\infty}[1+\sum_{\alpha=1}^{\infty}\frac{Q_\alpha}{t^\alpha}]t^{-v}u^{-t}dt, \qquad 0<u<1 \tag{3.8}$$

Also, since $t^{-v}[1+\sum_{r=1}^{\infty}Q_rt^{-r}] = 0(t^{-v})$, using Nair(1940), one can expand

$t^{-v}[1+\sum_{r=1}^{\infty}Q_rt^{-r}]$ in factorial series as

$$t^{-v}[1+\sum_{r=1}^{\infty}Q_rt^{-r}] = \sum_{\alpha=0}^{\infty}R_\alpha\frac{\Gamma[t+a]}{\Gamma[t+a+v+\alpha]} \tag{3.9}$$

where a is a constant to be determined later and coefficients R_α's can be determined explicitly as done below. Expanding logarithm of $\Gamma(t+a)/\Gamma(t+a+v+\alpha)$ using Barnes' expansion and converting back, one gets

$$\frac{\Gamma[t+a]}{\Gamma[t+a+v+\alpha]} = t^{-v-\alpha}[1+\sum_{j=1}^{\infty}C_{\alpha j}t^{-j}] \tag{3.10}$$

where

$$C_{\alpha j} = \frac{1}{j}\sum_{\ell=1}^{j}\ell A_{\alpha\ell}C_{\alpha,j-\ell}, \quad j=1,2,\dots \quad C_{\alpha 0} = 1 \tag{3.11}$$

and

$$A_{\alpha\ell} = \frac{(-1)^\ell}{\ell(\ell+1)}[B_{\ell+1}(a+v+\alpha) - B_{\ell+1}(a)] \tag{3.12}$$

Substituting (3.10) in (3.9) and comparing the coefficients of same powers of t on both sides, one gets

$$Q_\alpha = \sum_{j=0}^{\alpha}R_{\alpha-j}C_{\alpha-j,j}, \quad \alpha=1,2,\dots \quad R_0 = 1 \tag{3.13}$$

Now using (3.9) in (3.8) and integrating out term by term which is valid, the density of U is obtained as

$$f(u) = K^*(2\pi)^{\frac{1}{2}(pq-1)} p^{-(\frac{1}{2}mp+\delta p-\frac{1}{2})} \prod_{i=1}^{q} \gamma_i^{\gamma_i p(\frac{1}{2}m+\delta)-\frac{p(p+1)}{4}} s^v$$
$$\cdot \sum_{\alpha=0}^{\infty} R_\alpha u^{\frac{1}{2}ms+a-1} \frac{(1-u)^{v+\alpha-1}}{\Gamma(v+\alpha)}, \quad 0 < u < 1 \tag{3.14}$$

Since the series in (3.14) is uniformly convergent for $0 < u < 1$, the distribution function can be derived by integrating term by term, and this procedure would then lead to a series in incomplete beta functions given by

$$F(u) = K^* p^{-(\frac{1}{2}mp+\delta p-\frac{1}{2})} \prod_{i=1}^{q} \gamma_i^{\gamma_i p(\frac{1}{2}m+\delta)-\frac{p(p+1)}{4}} (2\pi)^{\frac{1}{2}(pq-1)} s^v$$
$$\cdot \sum_{\alpha=0}^{\infty} R_\alpha I_u(\frac{1}{2}ms+a, v+\alpha)\{\frac{\Gamma(\frac{1}{2}ms+a)}{\Gamma(\frac{1}{2}ms+a+v+\alpha)}\} \tag{3.15}$$

where $I_u(\cdot,\cdot)$ is the incomplete beta function. Also expanding K^* and

$\Gamma(\frac{1}{2}ms+a)/\Gamma(\frac{1}{2}ms+a+v+\alpha)$ using Barnes' expansion, one has

$$K^* = (2\pi)^{-\frac{1}{2}(pq-1)}(\frac{1}{2}m)^v p^{\frac{1}{2}mp+\delta p-\frac{1}{2}} \prod_{i=1}^{q} \gamma_i^{-\gamma_i p(\frac{1}{2}m+\delta)+\frac{p(p+1)}{4}}$$
$$\cdot[1+\sum_{\alpha=1}^{\infty} \frac{Q_\alpha^*}{(\frac{1}{2}m)^\alpha}] \tag{3.16}$$

where

$$Q_r^* = \frac{1}{r}\sum_{\ell=1}^{r} \ell A_\ell^* Q_{r-\ell}^*, \quad r = 1,2,\ldots \quad Q_0^* = 1 \tag{3.17}$$

$$A_r^* = -\frac{A_r}{s^r} \tag{3.18}$$

$$\frac{\Gamma(\frac{1}{2}ms+a)}{\Gamma(\frac{1}{2}ms+a+v+\alpha)} = (\frac{1}{2}ms)^{-v-\alpha}[1+\sum_{j=1}^{\infty} C_{\alpha j}(\frac{1}{2}ms)^{-j}] \tag{3.19}$$

and $C_{\alpha j}$'s are given by (3.11). Substituting (3.16) and (3.19) in (3.15) and simplifying we get the following result:

$$F(u) = I_u(\frac{1}{2}ms+a, v) + \sum_{\alpha=1}^{\infty} \frac{1}{(\frac{1}{2}m)^\alpha} G_\alpha \tag{3.20}$$

where

$$G_\alpha = \sum_{j=0}^{\alpha} R_{\alpha-j} I_u(\frac{1}{2}ms+a, v+\alpha-j) \sum_{\ell=0}^{j} \frac{Q_\ell^* C_{\alpha-j,j-\ell}}{s^{\alpha-\ell}} \tag{3.21}$$

Substituting $\alpha = 1, 2$ in (3.21) we get

$$G_1 = R_1 I_u(\frac{1}{2}ms + a, v+1))\frac{Q_0^*}{s}C_{10} + R_0 I_u(\frac{1}{2}ms + a, v)\{\frac{Q_0^*}{s}C_{01} + \frac{Q_1^*}{s^0}C_{00}\} \quad (3.22)$$

and

$$G_2 = R_2 I_u(\frac{1}{2}ms + a, v+2)\frac{Q_0^*}{s^2}C_{20} + R_1 I_u(\frac{1}{2}ms + a, v+1))\{\frac{Q_0^*}{s^2}C_{11} + \frac{Q_1^*}{s}C_{10}\}$$
$$+ R_0 I_u(\frac{1}{2}ms + a, v)\{\frac{Q_0^*}{s^2}C_{02} + \frac{Q_1^*}{s}C_{01} + \frac{Q_2^*}{s^0}C_{00}\} \quad (3.23)$$

From (3.13), (3.6), (3.11), (3.17) and (3.18) we have $R_0 = 1$, $Q_1 = R_1C_{10} + R_0C_{01}$, $Q_2 = R_2C_{20} + R_1C_{11} + R_0C_{02}$, $Q_0 = 1$, $Q_1 = A_1$, $Q_2 = \frac{1}{2}A_1^2 + A_2$, $C_{00} = C_{10} = C_{20} = 1$, $C_{11} = A_{11}$, $C_{01} = A_{01}$, $C_{02} = \frac{1}{2}A_{01}^2 + A_{02}$, $Q_0^* = 1$, $Q_1^* = A_1^*$, $Q_2^* = \frac{1}{2}A_1^{*2} + A_2^*$, $A_1^* = -\frac{A_1}{s}$, $A_2^* = -\frac{A_2}{s^2}$ and G_1 simplifies to

$$G_1 = \frac{1}{s}(A_1 - A_{01})\{I_u(\frac{1}{2}ms + a, v+1) - I_u(\frac{1}{2}ms + a, v)\}$$

where from (3.7) and (3.12) we get

$$A_1 = \frac{s}{2}\{\frac{1}{24}\sum_{i=1}^{q}\gamma_i^{-1}p(2p^2+3p-1) - \frac{1}{6p} - \delta[\frac{1}{2}pq(\gamma+1) - 1]\}$$

and $A_{01} = -\frac{v}{2}[2a + v - 1]$. Now if we choose $a = -\frac{1}{2}(v-1)$ and

$$\delta = \frac{p\prod_{i=1}^{p}\gamma_i^{-1}p(2p^2+3p-1) - 4}{12[pq(p+1) - 2]} \quad (3.24)$$

then $G_1 = 0$ and G_2 in (3.23) reduce to

$$G_2 = (-A_2^* - \frac{A_{02}}{s^2})\{I_u(\frac{1}{2}ms + a, v+2) - I_u(\frac{1}{2}ms + a, v)\}$$

where from (3.7), (3.18) and (3.12), we have

$$A_2^* = [\frac{1}{8}\delta^2(pq(p+1) - 2) - \frac{1}{192}p(p-1)(p+1)(p+2)\sum_{i=1}^{q}\gamma_i^{-2}] \quad (3.25)$$

and $A_{02} = \frac{v(1-v^2)}{24}$. In order to make $G_2 = 0$ we chose s^2 such that

$-A_2^* - \frac{A_{02}}{s^2} = 0$. Thus we obtain

$$s^2 = \frac{-v(1-v^2)}{24A_2^*}.$$

It may also be noted here that the choice of s is possible only if $v > 1$. For $q = 1$ and $p = 2$ we have $v = 1$ and in this case we choose $s = 1$. This case has already been studied by Kulp and Nagarsenker (1983) and can be dealt with separately. Finally simplifying G_3 in (3.12), from (3.20) we get the following asymptonic expansion of the distribution of U

$$F(u) = I_u(\frac{1}{2}ms + a, v) + \frac{R_3}{(\frac{1}{2}m)^3}\{I_u(\frac{1}{2}ms + a, v+3) - I_u(\frac{1}{2}ms + a, v)\} + 0(m^{-4})$$

It may be noticed that for $q = 1$ we get the results for the sphericity criterion derived by Kulp and Nagarsenker(1983).

4. ASYMPTOTIC NONNULL DISTRIBUTION

Consider the sequence of alternatives

$$A : I_p - \eta\Sigma_i^{-1} = \frac{2P_i}{\gamma_i m}, \quad i = 1, \ldots, q \tag{4.1}$$

where P_i, $i = 1, \ldots, q$ are fixed matrices as $m \to \infty$. Under this alternative, the h^{th} moment of U using Lemma 2.2 is given by

$$E(U^h) = \prod_{i=1}^{q} |I_p - \frac{2P_i}{\gamma_i m}|^{\gamma_i(\frac{1}{2}m+\delta)} \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{\kappa^{(1)}}\cdots\sum_{\kappa^{(q)}}\{p^{ph/s}/\prod_{i=1}^{q}\gamma_I^{\gamma_i ph/s}\}$$
$$\cdot\prod_{i=1}^{q}\left\{\prod_{j=1}^{p}\{\frac{\Gamma[\gamma_i(\frac{1}{2}m+\delta) + \frac{\gamma_i h}{s} - \frac{j-1}{2} + k_j^{(i)}]}{\Gamma[\gamma_i(\frac{1}{2}m+\delta) - \frac{j-1}{2}]}\}\frac{C_{\kappa^{(i)}}(2P_i/\gamma_i m)}{k^{(i)}!}\right\}$$
$$\cdot\frac{\Gamma[p(\frac{1}{2}m+\delta)+k]}{\Gamma[p(\frac{1}{2}m+\delta)+\frac{ph}{s}+k]}. \tag{4.2}$$

Using inverse Mellin transform the density of U is given by

$$f(u) = \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{\kappa^{(1)}}\cdots\sum_{\kappa^{(q)}} J(\kappa^{(1)}, \ldots, \kappa^{(q)})\Psi(u), \quad 0 < u < 1 \tag{4.3}$$

where

$$J(\kappa^{(1)}, \ldots, \kappa^{(q)}) = \prod_{i=1}^{q}\left\{|I_p - \frac{2P_i}{\gamma_i m}|^{\gamma_i(\frac{1}{2}m+\delta)}\frac{C_{\kappa^{(i)}}(2P_i/\gamma_i m)}{k^{(i)}!}\cdot\right.$$
$$\left.\{\prod_{j=1}^{p}\Gamma[\gamma(\frac{1}{2}m+\delta) - \frac{j-1}{2}]\}^{-1}\right\}\Gamma[p(\frac{1}{2}m+\delta)+k]. \tag{4.4}$$

and

$$\Psi(u) = \frac{1}{2\pi\imath}\int_{-\imath\infty}^{+\imath\infty} \{p^{ph/s}/\prod_{i=1}^{q}\gamma_i^{\gamma_i ph/s}\}$$
$$\cdot\{\prod_{i=1}^{q}\prod_{j=1}^{p}\Gamma[\gamma_i(\frac{1}{2}m+\delta)+\frac{\gamma_i h}{s}-\frac{j-1}{2}+k_j^{(i)}]\ //\Gamma[p(\frac{1}{2}m+\delta)+\frac{ph}{s}+k]\}$$
$$\cdot u^{-1-h}dh \tag{4.5}$$

Following the same procedure as in Section 3, we obtain the distribution of U in the following form

$$F(u) = (2\pi)^{\frac{1}{2}(pq-1)}s^v p^{-p(\frac{1}{2}m+\delta)-k+\frac{1}{2}}\prod_{i=1}^{q}\gamma_i^{\gamma_i p(\frac{1}{2}m+\delta)-\frac{1}{4}p(p+1)+k^{(i)}}$$
$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{\kappa^{(1)}}\cdots\sum_{\kappa^{(q)}\cdot}J(\kappa^{(1)},\ldots,\kappa^{(q)})\sum_{\alpha=0}^{\infty}R_\alpha(\kappa^{(1)},\ldots,\kappa^{(q)})$$
$$\cdot I_u(\frac{1}{2}ms+a,v+\alpha)\frac{\Gamma(\frac{1}{2}ms+a)}{\Gamma(\frac{1}{2}ms+a+v+\alpha)} \tag{4.6}$$

where the few coefficient $R_\alpha(\kappa^{(1)},\ldots,\kappa^{(q)})$ which we need are given below:

$$R_0(\kappa^{(1)},\ldots,\kappa^{(q)}) = 1$$

$$R_1(\kappa^{(1)},\ldots,\kappa^{(q)}) = \frac{s}{2p}[p\sum_{i=1}^{q}\gamma_i^{-1}a(\kappa^{(i)})-(k^2-k+p\delta k)]$$
$$R_2(\kappa^{(1)},\ldots,\kappa^{(q)}) = \frac{s^2}{6p^2}[-\frac{p^2}{4}\sum_{i=1}^{q}\gamma_i^{-2}\{a_2(\kappa^{(i)})-k^{(i)}\}-3\delta p^2\sum_{i=1}^{q}\gamma_i^{-1}a_1(\kappa^{(i)})$$
$$+\{k^3+(3\delta p-\frac{3}{2})k^2+(\frac{1}{2}+3\delta^2p^2-3\delta p)k\}]+\frac{1}{2}R_1(R_1+v+1)$$

Also, we have

$$\frac{\Gamma[p(\frac{1}{2}m+\delta)+k]}{\prod_{i=1}^{q}\prod_{j=1}^{p}\Gamma[\gamma_i(\frac{1}{2}m+\delta)-\frac{j-1}{2}]} = (2\pi)^{-\frac{1}{2}(pq-1)}p^{p(\frac{1}{2}m+\delta)+k-\frac{1}{2}}(\frac{1}{2}m)^{v+k}$$
$$\cdot\prod_{i=1}^{q}\gamma_i^{-\gamma_i p(\frac{1}{2}m+\delta)+\frac{1}{4}p(p+1)}\cdot[1+\frac{1}{(\frac{1}{2}m)}Q_1^*(k)+\{\frac{1}{(\frac{1}{2}m)}\}^2Q_2^*(k)+0(m^{-3})] \tag{4.7}$$

where

$$Q_1^*(k) = \frac{1}{2p}[k(2\delta p-1)+k^2]$$
$$Q_2^*(k) = -\frac{1}{6p^2}[(3p^2\delta^2-3p\delta+\frac{1}{2})k+(3p\delta-\frac{3}{2})k^2+k^3]+A_2^*+\frac{1}{2}Q_1^{*2}(k)$$

$$\prod_{i=1}^{q}|I-\frac{2P_i}{\gamma_i m}|^{\gamma_i(\frac{1}{2}m+\delta)}=\exp[-\sum_{i=1}^{q}trP_i]\Big[1+\frac{1}{m}\{-\sum_{i=1}^{q}(2\delta trP_i+\frac{trP_i^2}{\gamma_i})\}$$
$$+\frac{1}{m^2}\{-\sum_{i=1}^{q}(\frac{2\delta trP_i^2}{\gamma_i}+\frac{4trP_i^3}{3\gamma_i^2})$$
$$+\frac{1}{2}(-\sum_{i=1}^{q}(2\delta trP_i+\frac{trP_i^2}{\gamma_i}))^2\}+0(\frac{1}{m^3})\Big] \quad (4.8)$$

$$\frac{\Gamma[\frac{1}{2}ms+a]}{\Gamma[\frac{1}{2}ms+a+v+\alpha]}=(\frac{1}{2}ms)^{-v-\alpha}[1+\frac{C_{\alpha1}}{(\frac{1}{2}ms)}+\frac{C_{\alpha2}}{(\frac{1}{2}ms)^2}+\cdots] \quad (4.9)$$

and $C_{\alpha1}, C_{\alpha2}, \ldots,$ are given by (3.11) and (3.12). Substituting $R_0(\kappa^{(1)}, \ldots \kappa^{(q)})$, $R_1(\kappa^{(1)}, \ldots, \kappa^{(q)}), R_2(\kappa^{(1)}, \ldots, \kappa^{(q)})$, (4.7), (4.8) and (4.9) in (4.6), summing the resulting expression first over $\kappa^{(1)}, \ldots, \kappa^{(q)}$ using Lemma 2.1 and then over $\kappa = \{k^{(1)}, \ldots, k^{(q)}\}$, $k^{(1)} + \cdots + k^{(q)} = k$, using multinomial summation formula and finally over k, and neglecting higher order terms, we get the following result.

Theorem 4.1. Under the sequence of alternatives stated in (4.1), the nonnull distribution of U can be expanded asymptotically for large m as follows:

$$F(u)=I_u(\frac{1}{2}ms+a,v)+\frac{c}{m}[I_u(\frac{1}{2}ms+a,v)-I_u(\frac{1}{2}ms+a,v+1)]$$
$$+\frac{1}{m^2}\sum_{i=1}^{3}b_iI_u(\frac{1}{2}ms+a,v+i-1)+0(\frac{1}{m^3}) \quad (4.10)$$

where

$$c=\frac{1}{p}(\sum_{i=1}^{q}trP_i)^2-(\sum_{i=1}^{q}\frac{trP_i^2}{\gamma_i})$$
$$b_1=\frac{1}{2}c^2+2\delta c+\frac{4}{3}d,\quad d=\frac{1}{p}(\sum_{i=1}^{q}trP_i)^3-(\sum_{i=1}^{q}\frac{trP_i^3}{\gamma_i^2})$$
$$b_2=-c^2-\frac{4}{p}(\sum_{i=1}^{q}trP_i)c-\frac{2}{p}c-4\delta c$$

and

$$b_3=-(b_1+b_2)$$

Remark 4.1. For $q = 1$, the above result reduces to theorem 3.1 of Kulp and Nagarsenker(1983).

Now consider a more general alternative hypothesis given by

$$A : \begin{cases} I_p - \eta\Sigma_i^{-1} = 2P_i/\gamma_i m, & i = 1, \ldots, \ell \\ I_p - \eta^{-1}\Sigma_i = 2W_i/\gamma_i m, & i = \ell+1, \ldots, q \end{cases} \tag{4.11}$$

where $P_1, \ldots, P_\ell; W_{\ell+1}, \ldots, W_q$ are fixed matrices as $m \to \infty$. The asymptotic non-null distribution of U for this case is obtained by making the following replacements in the coefficients in (4.10)

$$tr P_i \to -tr W_i - \frac{2}{\gamma_i m} tr W_i^2$$

$$tr P_i^2 \to tr W_i^2 + \frac{4}{\gamma_i m} tr W_i^3$$

$$tr P_i^3 \to -tr W_i^3$$

for $i = \ell + 1, \ldots, q$. Thus we get the asymptotic distribution of U.

Theorem 4.2. Under the sequence of alternatives stated in (4.11), the non-nulll asymptotic distribution of U as $m \to \infty$ can be expanded as follows:

$$F(u) = I_u(\frac{1}{2}ms + a, v) + \frac{c^*}{m}[I_u(\frac{1}{2}ms + a, v) - I_u(\frac{1}{2}ms + a, v+1)] + \frac{1}{m^2}\sum_{i=1}^{3} b_i^* I_u(\frac{1}{2}ms + a, v + i - 1) + 0(\frac{1}{m^3}) \tag{4.12}$$

where

$$c^* = \frac{1}{p}(\sum_{i=1}^{\ell} tr P_i - \sum_{i=\ell+1}^{q} tr W_i)^2 - \sum_{i=1}^{\ell}\frac{tr P_i^2}{\gamma_i} - \sum_{i=\ell+1}^{q}\frac{tr W_i^2}{\gamma_i}$$

$$b_1^* = \frac{1}{2}c^{*2} + 2\delta c^* + \frac{4}{3}d^* - \frac{4}{p}(\sum_{i=\ell+1}^{q}\frac{tr W_i^2}{\gamma_i})(\sum_{i=1}^{\ell} tr P_i - \sum_{i=\ell+1}^{q} tr W_i) - 4\sum_{i=\ell+1}^{q}\frac{tr W_i^3}{\gamma_i^2}$$

$$d^* = \frac{1}{p^2}(\sum_{i=1}^{\ell} tr P_i - \sum_{i=\ell+1}^{q} tr W_i)^3 - \sum_{i=1}^{\ell}\frac{tr P_i^3}{\gamma_i^2} + \sum_{i=\ell+1}^{q}\frac{tr W_i^3}{\gamma_i^2}$$

$$b_2^* = -c^{*2} - \frac{4}{p}(\sum_{i=1}^{\ell} tr P_i - \sum_{i=\ell+1}^{q} tr W_i)c^* - \frac{2}{p}c^* - 4\delta c^* + \frac{4}{p}(\sum_{i=\ell+1}^{q}\frac{tr W_i^2}{\gamma_i})(\sum_{i=1}^{\ell} tr P_i - \sum_{i=\ell+1}^{q} tr W_i) + 4\sum_{i=\ell+1}^{q}\frac{tr W_i^3}{\gamma_i^2}$$

$$b_3^* = -(b_1^* + b_2^*).$$

Remark 4.2. Substituting $\ell = 0$ in the above expression we get the asymptotic nonnull distribution of U for the alternative

$$A : I - \eta^{-1}\Sigma_i = \frac{2W_i}{\gamma_i m}, \quad i = 1, \ldots, q$$

where $W_1, \ldots, W_q$ are fixed matrices as $m \to \infty$. Null distribution of U can be obtained by substituting $P_1 = \ldots = P_\ell = W_{\ell+1} = \ldots = W_q = 0$ in (4.12). The nonnull distribution of Mauchly's sphericity criterion can be obtained by substituting $q = 1$; $\ell = 0$ or $\ell = 1$ in asymptotic nonnull distribution of U given by (4.12), for the alternatives $(i) A : I - \eta\Sigma_1^{-1} = 2P_i/m$ and $(ii) A : I - \eta^{-1}\Sigma_1 = 2W_1/m$. The nonnull distribution of Neyman and Pearson criterion for testing equality of variances of univariate normal populatons can be obtained by substituting $p = 1$ in (4.12).

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