응용통계연구제 5권 2호
1992년 pp 169-179

Some Orthogonal Factorial Row-column Designs[1]

D. K. Park[2]

ABSTRACT

It is shown that a structurally complete row-column design has orthogonal factorial structure if each of its component designs has orthogonal factorial structure. It implies that such designs are most easily constructed via the amalgamating of one-dimensional block designs which have orthogonal factorial structure. However, this does not always hold for structurally incomplete row-column designs. A structurally incomplete row-column design is derived from the design with adjusted orthogonality, by simply interchanging row and treatment numbers.

1. Introduction

Row-column designs are those that involve two crossed, non-interacting blocking factors. A row-column design is called ***structurally complete*** (SC) by Stewart and Bradely (1990) if the design has no empty cells and ***structurally incomplete*** (SIC) otherwise. Many results for SC-designs, Latin Square designs being the simplest case, exist in the literature in their construction and properties. The construction and analysis of SIC-designs has been discussed recently by Stewart and Bradely (1991b), and Park and Dean (1992).

We restrict our attention to experiment in which v treatments (or treatment combinations) are observed r times each, and at most one treatment is observed at each combination of levels of the blocking factors. We will consider row-column designs with the same number of empty cells for each level of a blocking factor so that each ***component*** design (i.e. the design which is obtained by simply ignoring any one of the blocking factors) has constant block size. Suppose that a set of two component designs consists of two block designs such

---

1) This research was supported by the 1991-1992 Yonsei Maeji Research fund
2) Department of Statistics, Yonsei University, Wonju-kun, Kangwon-do, 222-701 Korea

that the i^{th} design has b_i blocks of size k_i $(=vrb_i-1;\ i=1,2)$. Since our designs have the same number of empty cells $(=b_1b_2-vr)$ for each blocking factor, vr should be divisible by b_i for all $i=1,2$ so that k_i's are positive integers.

A design is disconnected if the treatments can be split into groups such that no treatment from one group occurs in any blocks with any treatment from a different group. A design which is not disconnected is said to be ***connected***. The designs considered in this paper are all connected, and it is well known that every contrast of treatment effects in a connected design is estimable from within-block comparisons. The usual linear model is assumed to hold for all connected row-column designs considered in this paper

$$y = X_0\mu + X_1\beta_1 + X_2\beta_2 + X_3\tau + e \tag{1.1}$$

where y is a $vr \times 1$ vector observations, μ is a constant, τ is the vector of v treatment effects, β_i (i=1,2) is a vetor of effects of the b_i levels of the i^{th} blocking factor, and e is a vector of independent identically distributed random variables with mean zero and common variance σ^2. Following Stewart and Bradely (1991b), the information matrix C_d of row-column design d is

$$C_d = C_1 - MP^{-}M^{t} \tag{1.2}$$

where $M=N_2-k_1^{-1}N_1N_{12}$, $P=k_2I-k_1^{-1}N_{21}N_{12}$, $N_1=X_3^tX_1$, $N_2=X_3^tX_2$, $N_{12}=N_{21}^t=X_1^tX_2$, and P^- is a generalized inverse of P.

Once the effects of the treatment combinations have been adjusted for blocks, the estimates (main effects and interactions) may now be correlated, so that interpretation of the analysis becomes difficult. However, this problem can usually be solved since there are a large number of block designs with ***orthogonal factorial structure*** (OFS). A factorial design is said to have OFS if the treatment sum of squares adjusted for block effects admits an orthogonal splitting into components corresponding to different factorial effects, so that any effect can be estimated and assessed independently of any other effect. Several methods of constructing SC designs with OFS have been developed based on Kronecker product (see Gupta, 1983) and by the use of a generalized cyclic design (see John and Lewis, 1983).

2. SC-design with OFS

Following Mukerjee(1979), for an n-factor experiment with factor F_i at m_i levels (i=1,⋯,n), the treatment structure is given by

$$\tau = \sum_{X} S_X \tau \qquad (2.1)$$

where the summation is over all binary numbers $x=(x_1,\cdots,x_n)$ and

$$S_X = S_{X1} \otimes S_{X2} \otimes ---- \otimes S_{Xn} ,$$

where ⊗ denotes Kronecker product and

$$S_{x_j} = \begin{cases} m_i^{-1} J_{m_i} & \text{if } x_i = 0 \\ I_{m_i} - m_i^{-1} J_{m_i} & \text{if } x_{i=1} \end{cases}$$

I_n is an n × n identity matrix and J_n is an n × n unit matrix.

In Theorem 2.1, we show that a d-dimensional factorial SC design (especially when d=2, row-column) has OFS whenever each component design has OFS.

Lemma 2.1 (Mukerjee, 1979) For a block design, OFS holds if and only if the information matrix commutes with S_x for every $x \neq 0$.

A design is called to be ***completely symmetric*** if information matrix of the design has the form of $aI_v + bJ_v$. Such designs satisfy the condition of Lemma 2.1 and have OFS. Thus, randomized complete block designs and balanced block designs are examples of one-dimensional designs with OFS.

Theorem 2.1 A factorial SC-design has OFS whenever each component design has OFS.

Proof. Cheng (1978) has shown that the information matrix C_d of an SC-design d can be expressed as

$$C_d = \sum_{h=1}^{d} C_h - r(d-1)(I - v^{-1}J) \qquad (2.2)$$

where C_h is the information matrix of the h^{th} component design. Assume that each of d component designs has OFS. Then Lemma 2.1 implies that $C_hS_x = S_xC_h$ for h=1,⋯,d and so $\sum_{h=1}^{d} C_hS_x = \sum_{h=1}^{d} S_xC_h$. Therefore, from (2.2)

$$\{\ C_d + r(d-1)(I-v^{-1}J)\ \}\ S_x = S_x\ \{\ C_d + r(d-1)(I-v^{-1}J)\ \} \tag{2.3}$$

Note that since all S_x matrices, apart from S_0, are contrast matrices and symmetric, they are proper matrices, so that

$$S_xJ = JS_x = 0 \tag{2.4}$$

Hence, from (2.3) and (2.4), $C_dS_x = S_xC_d$ for all x. Thus, the d-dimensional SC-design has OFS from Lemma 2.1.

The theorem tells us that Latin square designs, Youden designs, generalized Youden designs, Pseudo-Youden designs, generalized cyclic designs, and Youden hyperrectangles are examples of two- or higher-dimensional orthogonal factorial SC-designs with component designs having OFS. Those designs are most easily constructed via the ***amalgamating*** of one-dimensional block designs which have OFS.

3. A class of SIC-designs with OFS

Theorem 2.1 guarantees that an SC-design has OFS, provided that each of its component design has OFS. However, even if each of the component designs has OFS, the corresponding SIC-design might not have OFS, as shown by the following example.

Example 3.1 Take two SIC-designs with v=4 and r=3 with treatment labels 00, 01, 10, 11 as follows. In the design, x denotes the cell is empty.

00	01	x	x
01	x	11	x
10	x	x	00
x	11	10	x
x	00	x	11
x	x	01	10

Two component designs have OFS since they are balanced incomplete block designs. However, since $S_{10}C_d \neq C_dS_{10}$ where $S_{10} = (I_2-0.5J_2)\otimes(0.5J_2)$, we can conclude that this SIC-design does not have OFS, even if the two component design have OFS.

In this paper, we will construct a class of completely symmetric SIC designs that have OFS. In the equation (1.2) if M=0, C_d is same to C_1. Estimates of treatment parameters, therfore, are the same to those obtained from a model in which the column parameters have been deleted. These row-column designs will be said to be column-orthogonal. It implies that if its row-component design is completely symmetric, the column-orthogonal SIC design is completely symmetric and has OFS. In section 4, we construct such kind of designs.

4. Construction of column-orthogonal designs with OFS

Let $N_{12}(\mu,\tau)$ denote the row x column incidence matrix in the reduced normal equation after adjusting for μ and τ. Following Eccleston and Russell (1975), blocking factors adjusted for τ are orthogonal if and only if $N_{12}(\mu,\tau)$ is the same whether or not β_j is included in the model (j=1,2). Corollary 2 of Eccleston and Russell (1975) implies that a design δ has adjusted orthogonality if and only

$$N_{12} = (r)^{-1}N_1^tN_2 \tag{4.1}$$

For an SC-design δ, the equation (4.1) becomes

$$J_v = (r)^{-1}N_1^tN_2 \tag{4.2}$$

An SIC-design can be derived from the design δ with parameters v, b_1, b_2 and r, by simply interchanging row and treatment numbers. Such ideas were originated by Agrawal (1966) and developed by Stewart and Bradely (1991b). The resulting design so constructed, say ω, has $v(\omega)=b_1$, $b_1(\omega)=v$, $b_2(\omega)=b_2$, $k_1(\omega)=r$. Let the corresponding incidence matrices of the design ω be $N_1(\omega)$, $N_2(\omega)$, and $N_{12}(\omega)$. Then the equation (4.1) becomes

$$k_1 N_2(\omega) = N_1(\omega) N_{12}(\omega) \tag{4.3}$$

and M=0, so the design ω is column-orthogonal.

However, such designs are not necessarily to be completely symmetric. An example will make clear this point. Take an SC-design δ that was given by John and Eccleston (1986) with v=12, b_1=4, b_2=6 and r=2 : for simplicity, each treatment combinations is represented by the numbers 0,...,12.

0	4	8	3	7	11
1	5	9	0	4	8
2	6	10	5	9	1
3	7	11	10	2	6

We get the following SIC column-orthogonal design ω with $v(\omega)=4$, $b_1(\omega)=12$, $b_2(\omega)=6$, $k_1(\omega)=2$, $k_2(\omega)=4$, and $r(\omega)=6$ by arranging the treatments in the rows and the row blocking labels in treatments.

0	x	x	1	x	x
1	x	x	x	x	2
2	x	x	x	3	x
3	x	x	0	x	x
x	0	x	x	1	x
x	1	x	2	x	x
x	2	x	x	x	3
x	3	x	x	0	x
x	x	0	x	x	1
x	x	1	x	2	x
x	x	2	3	x	x
x	x	3	x	x	0

The above resulting design ω is column-orthogonal but not completely symmetric. We define three special classes of SIC-designs amalgamated from binary component block designs. Designs with OFS are selected in each class.

Definition 4.1 An SIC-design in which one component design is a complete block design and the second component design is a balanced block design is called a ***Youden Square Type*** design.

Definition 4.2 A ***Latin Square Type*** SIC-design is a square matrix of the treatment labels such that each treatment labels occurs in each row exactly once and in each column exactly once.

Definition 4.3 An SIC-design is defined to be a ***generalized Youden Square Type*** if two component designs are both balanced designs.

Whether the resulting design ω is whether completely symmetric or not is determined by the structure of row component design of the selected design δ. If every block of row component designs has a constant number of treatment labels in common with every other block of the component design, the resulting design ω should be completely symmetric. We can see that many of the row-column designs with adjusted orthogonality in the literature satisfy the above condition (for example, Park and Dean (1990)) and we can construct an SIC-design with OFS from the existing design with adjusted orthogonality, by interchanging row and treatment numbers. Some examples are shown below.

The design ω in Example 4.1 with parameters $v(\omega)=b_1=5$, $b_1(\omega)=v=10$, $b_2(\omega)=b_2=6$, and $k_1(\omega)=r=3$ is a column-orthogonal and completely symmetric Youden Square Type design.

Example 4.1

(a) SC-design δ

1	0	9	5	4	3
5	6	3	2	7	0
7	4	6	3	1	8
9	5	4	8	2	7
6	8	2	1	0	9

(b) derived 'Youden Square Type' ω

x	0	x	x	4	1
0	x	x	4	2	x
x	x	4	1	3	x
x	x	1	2	x	0
x	2	3	x	0	x
1	3	x	0	x	x
4	1	2	x	x	x
2	x	x	x	1	3
x	4	x	3	x	2
3	x	0	x	x	4

A Latin Square Type SIC-design ω can be constructed from Youden Square design δ. Since the design δ has $v=b_2$ and $r=b_1$ and so the design ω derived from the method has $b_1(\omega)=b_2(\omega)=r(\omega)$ and $k_1(\omega)=k_2(\omega)$. The following example with parameters $v(\omega)=b_1=4$, $b_1(\omega)=v=7$, $b_2(\omega)=b_2=7$ and $k_1(\omega)=r=4$ serves as an illustration.

Example 4.2

(i) Youden Square δ

1	6	5	2	3	4	7
2	7	6	4	5	3	1
3	2	1	5	6	7	4
4	3	2	7	1	5	6

(ii) derived Latin Square Type ω

0	x	2	x	3	x	1
1	2	3	0	x	x	x
2	3	x	x	0	1	x
3	x	x	1	x	0	2
x	x	0	2	1	3	x
x	0	1	x	x	2	3
x	1	x	3	2	x	0

Design (ii) is column-orthogonal and since $C_w=C_1(=C_2)=7I-(4/7)J$ it is completely symmetric. Note that all Latin Square Type designs are column-orthogonal and completely symmetric since $N_1=N_2=J$ and so $M=0$ in the equation (1.2).

A generalized Youden Square Type design ω can be derived from SIC-design δ which was constructed by Park and Dean (1992). The following completely symmetric generalized Youden Square Type design with parameters $v(\omega)=5$, $b_1(\omega)=10$, $b_2(\omega)=5$ and $k_1(\omega)=2$ is derived from the design δ in Example 3 of Park and Dean (1992).

Example 4.3

x	x	x	1	0
x	x	0	x	3
x	0	3	x	x
2	x	x	x	1
1	x	3	x	x
x	3	x	x	2
x	x	1	3	x
3	3	x	x	x
x	2	x	0	x
3	x	x	2	x

Acknowledgement

I appreciate referees' helpful editing of this paper.

References

[1] Agrawal, H.L.(1966). "Some Methods of Construction of Designs for Two-way Elimination of Heterogeneity," *Journal of American Statistical Association*, 61, 1153-1171.

[2] Cheng, C.S.(1978). "Optimal Design for the Elimination of Multiway Heterogeneity," *Annals of Statistics*, 6, 1262-72.

[3] Eccleston, J.A. and Russell, K.G. (1975). "Connectedness and Orthogonality in Multi-factor Designs," *Biometrika*, 62. 341-345.

[4] Gupta, S.C. (1983). "Some New Methods for Constructing Block Designs having Orthogonal Factorial Structue," *Journal of Royal Statistical Society*, B, 45, 297-307.

[5] John, J.A. and Eccleston, J.A. (1986). "Row-column α-designs," *Biometrika*, 73, 301-306.

[6] John, J.A. and Lewis, S.M. (1983). "Factorial Experiments in Generalized Cyclic Row-column Designs," *Journal of Royal Statistical Society*, B, 45, 245-251.

[7] Mukerjee, R. (1979). "Inter-effect-orthogonality in Factorial Experiments," *Calcutta Statistical Association Bulletin*, 28, 83-108.

[8] Park, D.K. and Dean, A.M. (1990). "Average Efficiency Factors and Adjusted Orthogonality in Multi-dimensional Designs," *Journal of Royal Statistical Society*, B, 52, 361-368.

[9] Park, D.K. and Dean, A.M. (1992). "Structurally Incomplete Row-column Designs with Adjusted Orthogonality," *Communications in Statistics- Theory Methods*, B, 21(12) (to appear).

[10] Stewart, F.P. and Bradely, R.A. (1990). "Some Adjusted Orthogonal, Variance Balanced, Multidimensional Block Designs," *Communications in Statistics-Theory Methods*, B, 19(6). 2109-2144.

[11] Stewart, F.P. and Bradely, R.A. (1991a). "Some Universally Optimal Row-column Designs with Empty Nodes," *Biometrika*, 78, 337-348.

[12] Stewart, F.P. and Bradely, R.A. (1991b). "Intrablock Analysis of Designs with Multiple Blocking Criteria," *Journal of American Statistical Association*, 86, 792-797.

직교 요인 행-열 계획[1)]

박 동 권[2)]

요 약

블록 계획에서 요인의 효과를 독립적으로 추정 할 수 있도록 그 효과를 직교되게 구성 했을 때 그 계획을 직교 요인 계획이라 한다. 빈 칸이 없는 요인 행-열 계획의 성분계획들이 직교된 요인의 성질을 지니면 그 행-열 계획 또한 직교된 요인의 성질이 유지 된다는 사실이 입증되었다. 이러한 이유로 빈 칸이 없는 직교요인 행-열 계획은 직교된 성질을 가지는 일차원 성분 계획들을 단순히 결합함으로서 만들 수 있다.

그러나, 이러한 관계는 빈 칸이 있는 요인 행-열 계획에서는 성립되지 않는다. 빈 칸이 있는 요인 행-열 계획의 연구는 근래에 와서 그 필요성으로 활발히 진행되고 있다. 본 논문에서는 직교된 요인의 성질을 갖는 빈 칸이 있는 요인 행-열 계획의 일부를 조정 직교된 행-열 계획에서 행과 처리를 바꿈으로 설계해 보았다.

---

1) 이 논문은 1991-1992년도 연세대학교 매지 학술연구비에 의하여 지원받았슴.
2) (222-701) 강원도 원주군 흥업면 연세대학교 문리대학 통계학과 조교수