Computation of Inelastic Deflection of Slab by Elastic Finite Element Analysis

탄성 유한요소 해석에 의한 슬래브의 비탄성 처짐 산정

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요 약

사용하중하의 철근콘크리트 스래브의 비탄성 처짐을 산정하는 실용적인 방법을 제시하였다. 선형 유한요소 해석의 탄성해석결과와 설계된 철근량을 이용하여 비탄성 처짐계수(β)를 결정하고, 이를 이용하여 설계된 슬래브의 사용성을 검토하기 위한 사용하중하의 처짐을 구할 수 있도록 하였다. 모서리에 지지된 슬래브 예에서 제시한 방법으로 구한 비탄성처짐과 실험 및 비선형해석 결과와 비교해 본 결과 서로 매우 잘 일치함을 보여 주었다. 제시된 방법을 비정형 슬래브 설계에 응요한 문제도 고려하였다.

Abstract

A practical method of estimating inelastic deflection of reinforced concrete slab under service load is presented. Based on the elastic results of linear finite element analysis and area of reinforcement, inelastic deflection multiplier (β) is evaluated and desired deflection as a measure of serviceability of the designed slab is obtained. Example for the corner supported slab shows that the results from the proposed method agree well with those from the experiment/and nonlinear finite element analysis. Application of the method to the design of irregular slab is also considered.

1. INTRODUCTION

Evaluation of appropriate deflections as a measure of the serviceability of the structures is one of the important parameters in the design of reinforced concrete slab systems. However the calculation of deflection in two-way reinforced concrete slab systems is not a straightforward matter. The difficulties arise due to various boundary conditions, irregular shape of the slab, or various loading conditions. In addition to the geometric conditions, reinforced concrete is materially nonhomogeneous and it develops cracks. Furthermore, arrangement of steel bars are different all over the floor, depending on its flexural characteristics.

Classical elastic plate theory provides solutions for relatively simple boundary and loading conditions(Timoshenko, 1959 and Szilard, 1974). And approximate solutions for the regular slab can be obtained by the equivalent frame analysis (Kripanarayanan, 1976). However, these methods are applicable only to the limited conditions. The finite element method is considered to be one

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이 논문에 대한 토론을 1992년 6월 30일까지 본 학회에 보내주시면 1992년 12월호에 그 결과를 내재하겠습니다.

of the more appropriate approach for this problem, since the accurate solutions can be obtained for the slab of any shape or boundary conditions. But the elastic deflections are not acceptable for use in the serviceability check of the designed slab, since they do not consider the effects of concrete cracking and reinforcement on the member stiffness as stipulated in the ACI 318 code(ACI, 1989). These complicated characteristics can be incoporated by using the nonlinear finite element method. However, the application of the method to the practical slab design problem is not viable, due to the complexity of modeling and expensive solution cost. Moreover, proper modeling for reinforcement prior to nonlinear analysis is another problem in the design process, since the steel areas can not be determined until the analysis is completed. Hence a practical method for use in slab design to evaluate service load deflection combined with linear finite element method is necessary.

2. ELASTIC MOMENTS OF THE SLAB

The moment values(M_x, M_y) of the slab required for succeeding calculations are obtained from the finite element analysis of the slab. In this analysis isoparametric plate bending element is used. For the element shown in the Fig.1, simple expression of the x and y direction moments can be given as(Lee, 1987).

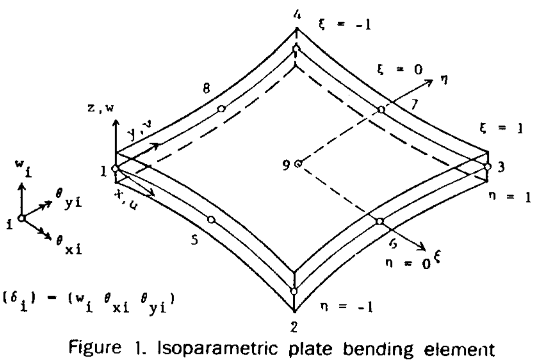


Figure 1. Isoparametric plate bending element

$$\begin{Bmatrix} M_x \\ M_y \end{Bmatrix} = D\sum_{i=1}^{N} \begin{Bmatrix} \alpha_i\theta_{yi} - \nu\beta_i\theta_{xi} \\ \nu\alpha_i\theta_{yi} - \beta_i\theta_{xi} \end{Bmatrix} \tag{1}$$

where

D =flexural rigidity of slab
$=Et^3/12(1-\nu^2)$ in which 'E' is Young's modulus, 't' is thickness of the slab, and 'ν' is Poisson's ratio

N= number of nodes of an element

θ_{yi} $=\theta_y$ rotation ot node i

θ_{xi} $=\theta_x$ rotation ot node i

In the above expression α_i and β_i are given as

$$\begin{Bmatrix} \alpha_i \\ \beta_i \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\partial \Psi_i}{\partial \xi} \\ \frac{\partial \Psi_i}{\partial \eta} \end{Bmatrix} \tag{2}$$

where

$\xi,\eta=\xi,\eta$ coordinate of isoparametric element

Ψi=shape function at node i

In addition to the moment values in both directions, the areas of the steel are also necessary to compute transformed steel area of the reinforced slab. In the analysis problem, they should be provided prior to the analysis. However, in the design problem, the steel areas must be calculated using the design governing moments of the elastic solution and the final designed values are obtained by comparing the minimum code requirement.

3. INELASTIC DEFLECTION MULTIPLIER(β) METHOD

In the elastic theory, deflection of the isotropic beam can be expressed as

$$w = k\frac{q\ell^4}{EI} \tag{3}$$

where

w =deflection in the direction of loading

k =factor depending on loading pattern and boundary condition

q =uniformly distributed load

ℓ =span length

I =moment of inertia of the beam section

The above expression assumes pure elastic condition, but it should be modified if the effects of concrete cracking are to be considered. Branson(1965) developed the effective moment of inertia approach in the beam deflection and ACI code adopted this method since 1971. In this method, the gross moment of inertia, I, is reduced to an effective moment of inertia, I_e, taking into account the cracked transformed section of the beam. Then inelastic deflection, Δ, can be expressed as

$$\Delta=k\frac{q\ell^4}{EI_e} \tag{4}$$

Dividing Eq. (4) by Eq. (3), the ratio, β, which is referred to as inelastic deflection multiplier is obtained.

$$\beta=\frac{\Delta}{w}=\frac{I}{I_e}\geq 1.0 \tag{5}$$

Note that in Eq (5), Young's modulus E is assumed to be the same before and after cracking. Rewriting Eq. (5) in terms of Δ

$$\Delta=\beta w \tag{6}$$

Thus final deflection Δ is obtained simply multiplying β to the elastic deflection w. If one considers unit width of the slab in each direction, similar relationships can be obtained for the two-way slab. The β value of the slab in x direction similar to Eq. (5) can be expressed as

$$\beta_x=\frac{I}{I_{ex}} \tag{7}$$

where I_{ex} is effective moment of inertia about y axis considering unit width of the slab in y direction, and it can be expressed as

$$I_{ex}=(M_{cr}/M_x)^3 I_g+[1-(M_{cr}/M_x)^3]\ I_{cr,x}\leq I_g \tag{8}$$

where

M_{cr} $=f_r I_g/y_t$=cracking moment

f_r =modulus of rupture of concrete $=0.65\ \sqrt{W_c f_c'}$ where W_c is the unit weight of concrete and f_c' is strength of the concrete

y_t =distance from extreme tension fiber to neutral axis of gross uncracked section

M_x =aplied moment per unit width in x direction at the location under which deflection is computed. It is given in Eq. (1).

I_g =moment of inertia of gross uncracked section of unit width, neglecting reinforcement.

$I_{cr,x}$ =moment of inertia of transformed cracked section considering unit width of slab in y direction

Similar expression for β_y can be obtained in y direction, By averaging β_x and β_y, the final β value of the slab at the desired location is then obtained.

To compute service load deflections, β values are necessary for the loading condition of unfactored loads. However, they can be evaluated only when steel areas are known, since moment of inertia of transformed cracked section involve steel areas. In the analysis problem, the steel areas are given prior to the analysis, hence β values are computed whenever analysis of certain load case is done and resulting moments for that load case are available. But in the design problem, the steel areas are determined at the last stage of analysis, namely after obtaining governing design moments from the analysis of several different cases of factored loading conditions. Thus β values for service load condition can be estimated only when all the strength design process is completed.

Once β values are obtained, service load deflections can be evaluated multiplying the elastic deflections by the corresponding β values at the desired nodal points. Actually, in the design problem, only the maximum deflection needs to be checked for serviceability criterion.

4. COMPARISON WITH EXPERIMENT AND NONLINEAR ANALYSIS

To see how the β-method provides reasonable solutions, an analysis problem is considered. Corner supported slab under a central point load tested by McNeice(1971) is shown in Fig.2. Due to the symmetry conditions one quarter model is considered. A total of 36, 8-node elements are used and it is shown is the Fig.2 at upper right sector. Since the meshes are rectangular and thickness to width ratio is small(1/21), reduced integration rule (2×2 points in Gauss quadrature) is applied to the element stiffness formulation of bending and shear term.

β values for each load case are computed using pre-assigned steel areas and moment results of corresponding load case. The contour plot of β values for load p=3.2 kips is shown in Fig.3, and these values are used to evaluate deflections in the load-deflection curves. As shown in the Fig.3, β has its greatest value (around 5) at center of the slab upon which concentrated load is applied. Which means that at the center of the slab, inelastic deflection considering the effects of concrete cracking and steel areas is about 5 times greater than the elastic deflection. Inelastic deflections at other places can be estimated in the similar manner.

The contour plots for different loads are also obtained and it was observed that contour line of high β value moves toward the corner as load increases. This indicates that inelastic region is expanded with higher loads.

Comparison of load-deflection curves at node 21 and 61(see Fig.2) are shown in Figs.4 to 5 . If service load is assumed as twice the slab weight(0.38 kips), the deflections computed from β-method shows a good agreement with those obtained from experiment and nonlinear finite element analysis up to 5 to 6 times the service load level. Beyond this load level, solutions at the central part of the slab, which represent maximum deflections, are still good all the way up to failure(Fig.4), whereas the solutions are getting stiffer when the locations are away from the center of the slab(Fig.5). In this example, many load cases are analyzed to obtain corresponding points in the load-deflection curves, but it is not necessary to do this if one disires a solution at certain load level. The analysis is required only once for this load level and corresponding β value is computed. Then desired inelastic deflection is obtained directly from the elastic deflection. This is a big difference between the β-method and nonlinear finite element method in which all the previous solutions must be obtained to get the solutions at certain load level. Thus the latter requires much more computational efforts and makes the approach almost impractical for design application.

In view of this example and other examples (Lee, 1987), it is concluded that the method combined with linear finite element analysis provides a very good solution at the service load level, and it produces a good approximation up to 5 to 6 times service load level. Thus the method is considered to be a very practical tool for evaluating the service load deflection in the reinforced concrete two-way slabs, taking into

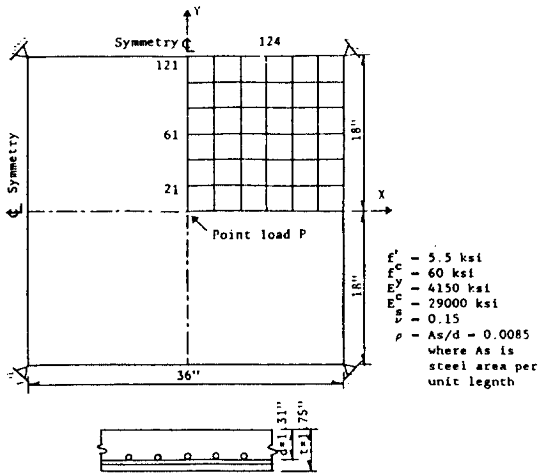
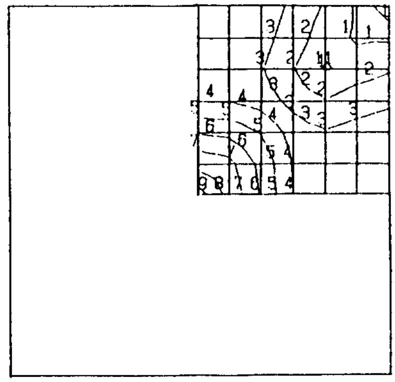


Figure 2. Corner supported two-way slab tested by McNeice.



Contour	1	2	3	4	5	6	7	8	9
β	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0

Figure 3. β contour for load P=3.2 kips.

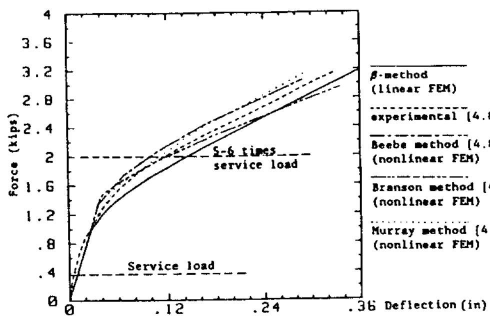


Figure 4. Load-deflection curves at node 21.

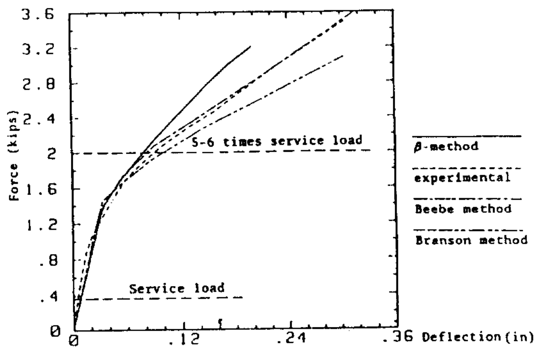
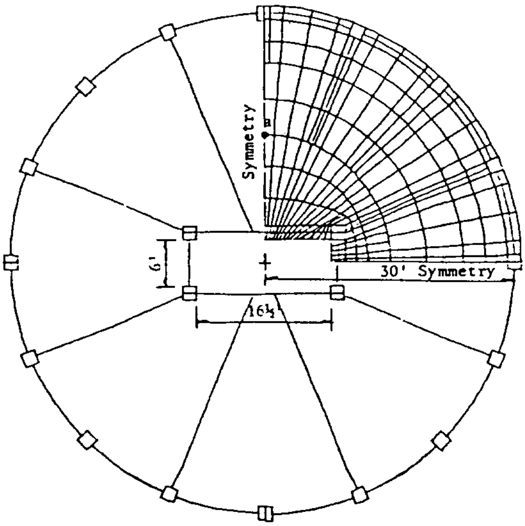


Figure 5. Load-deflection curves at node 61.

account the effects of cracking and reinforcement. Application of the β-method to the design problem is a little different from that to the analysis problem. As mentioned earlier, in design problem, the steel areas must be estimated first to proceed the method.

To show how the method is applied to practical disign problems, a concrete slab of irregular shape is considered. The plan and a quarter model of finite element mesh is shown in Fig. 6. As shown in the Fig.6, the circular slab has rectangular opening at its central part. Due to the complexity of geomertry, other means of analysis except finite element method is almost impossible. After analyzing the structure for the several combinations of load cases, elastic deflections and the moments in radius and circular directions are obtained. In Fig.7, deformed shape of a quarter model under the service load condition is shown.



Story height - 9 ft
Thickness of slab - 7 in
All columns - 20 x 20 in
All beams - 15 x 25 in
f'_c = 4 ksi for slab, beam, and column
f_y - 60 ksi
Superimposed dead load - 40 psf
Live load - 100 psf

Figure 6. Plan and quarter model of circular slab with opening.

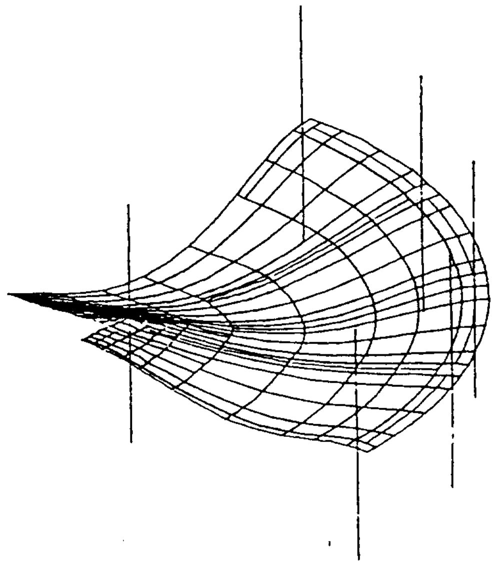
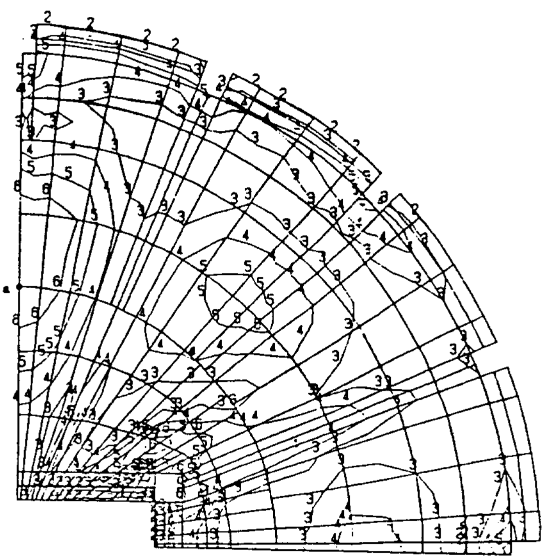


Figure 7. Deformed shape of circular slab.



Contour	1	2	3	4	5	6	7
β	-2.00	0.00	2.00	4.00	6.00	8.00	10.00

Figure 8. β contour for the circular slab.

Using the disign governing moments taken from computed values of all load cases, the required steel areas for the desired locations are computed. Then those values are compared with ACI code minimum steel area to determine final designed steel areas.

Once the steel areas are determined, the β values can be computed according to the procedure described above. Based on these values, the contour plots of β for the service load conditions are obtained. To compare the maximum ACI code limit on live load deflections as serviceability check, the β values for unfactored dead load and unfactored dead load plus live load should be determined. The former is shown in Fig.8.

From the Fig.7, it is observed that the maximum deflection occurs at mid span(point 'a' in Fig.6). Therefore, the maximum inelastic service load deflection can easily be obtained by multiplying β at this location to the elastic deflection. The maximum service load deflection together with the ACI code limit are shown in the Table 1. In this example, all the strength

Table 1. Maximum service load deflection (in.) at 'a' in Fig.6.

Deflection	DL only (Δ_{DL})	DL+LL (Δ_{DL+LL})	LL (Δ_{LL})	Code limit (L/360)
Elastic	0.1092	0.1836		
Multiplier	9.16	11.16		
Inelastic	1.00	2.05	1.05	0.87

criteria for the moment and the shear are satisfied. However, the designed slab violates the serviceability criteria as shown in the Table.1 Therefore due consideration must be given and redesign should be performed. Possible adjustment may be either increase slab thickness or provide circular beam at mid-span to reduce the deflection.

5. CONCLUSIONS

A practical means of estimating service load deflection of the concrete slab is presented. The mothod is considered to be very useful tool in measuring serviceability of the slab. From the observations and discussions presented in the paper, the following conclusions can be drawn:

1. Inelastic deflection multiplier (β) is evaluated utilizing the elastic results from the linear finite element analysis in which isoparametric plate bending element is used.

2. In computing β value, the effect of reinforcement and cracking of concrete is duly considered by introducing the effective moment of inertia approach to the slab. Hence appropriate inelestic deflections are obtained by multiplying β to the elastic deflections.

3. The comparison with the results from experiment and nonlinear finite element analysis for the corner supported slab shows that the β-mothod provides a realistic solution at the service load level without involving any nonlinear process.

4. The method is believed to be particularly suitable for the design application, since the steel areas are disigned in advance through the strength design process by elastic analysis, then using the service load deflection the serviceability is duly checked for the designed slab.

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(접수일자 : 1991. 8.13)