論文 한국정밀공학회지 제9권 제2호 1992. 6
Journal of the Korean Society of Precision Engineering Vol. 9. No. 2 Jun. 1992.

경계요소법에 의한 이종재료내 크랙의 응력확대계수 평가

김 상 철*, 임 원 균**

Evaluation of the Stress Intensity Factor for a Crack in Bimaterial Plate by the Boundary Element Method

Sang-Chul Kim*, Won-Kyun Lim**

초 록

이종재료의 접합면에 수직으로 존재하는 크랙에 대하여 경계요소 해석을 수행하여, 그결과 실용가능한 수치 근사해를 얻을 수 있었다. 크랙을 정확히 모델링하기 위하여 크랙표면을 분리영역으로 하는 영역분할법을 채택하였으며, 해의 정확성을 향상시키기 위하여 등매개 2차요소로의 경계분할과 함께 크랙선단에서 표면력의 특이성을 나타내도록 하였다. 응력확대계수는 크랙표면상 절점의 상대변위를 이용하여 결정하였다. 또한 이종재료내 크랙에 대하여 응력확대계수를 간단히 구할수 있는 간편해석법을 제안하고 이의 적용 가능한 범위를 제시하였다.

Nomenclaure

x, y	: cartesian coordinates	Γ	: boundary of the body
2a	: crack length	u_j	: displacements components at boundary
2w	: plate width of first material containing a crack	p_j	: tractions components at boundary
b	: plate width of second material	U_{ij}, P_{ij}	: components of the tensors corresponding to the fundamental solutions
t	: plate thickness	N	: shape function
E_1	: Young's modulus of first material containing a crack	η	: local coordinate
E_2	: Young's modulus of second material	J	: Jacobian
α	: dimensionless inertia parameter (E_2I_2/w^3t)	L	: length of crack tip element
β	: dimensionless extensional rigidity (E_2b/E_1w)	G	: shear modulus
I_2	: inertia moment of area of second material	ν	: Poisson's ratio
K_I, K_{II}	: mode I and mode II stress intensity factors	κ	: $3-4\nu$ in plane strain, $(3-\nu)/(1+\nu)$ in plane stress
F	: correction factor for stress intensity factor ($F=K_I/\sigma\sqrt{a}$)	σ	: normal stress of plate
		P	: applied load
		ε	: normal strain

* 인하대학교 기계공학과(정회원)
** 명지대학교 기계공학과

1. INTRODUCTION

The use of composite materials as structural components has inspired considerable research[1] on the effect of flaws and imperfections on the structural strength. The appearence of a flaw or crack in material could lead to a mode of failure where fracture occurs with nominal cross-section stresses well below the material yield stress. As the behavior of cracks in plates composed of two different materials bonded together is fundamental to the whole behavior of composite materials, the problems of crack in a bimaterial plate should be studied.

The boundary element methods have considerable advantages for such problems since cracks may be accurately modeled by considering only the boundary of the problem geometry. There are two classes of boundary element methods : indirect[2-3] and direct.[4-5]. The indirect method gives poor resolutions of stresses near the crack surface, which is the region of major importance in fracture mechanics. On the other hand, the direct method produces a singular algebraic system for an idealized crack. The direct method is therefore commonly used for problems with cracks, and we adopt it in the present work.

We have already described in the literature[6] the application of boundary element method for the prediction of crack growth in two dimensional crack problems, based on sub-region partitioning along crack lines,[7] and this approach is now extended to study the stress intensity factors for a crack in two dimensional plate, with crack running normal to a bimaterial interface.

The simple analytic method is also presented to obtain the stress intensity factors for a crack in bimaterial plate. It is based on the transformed section corresponding to an equivalent plate made of the second material. Numerical and analytic results are given for the stress intensity factors of mode I for cracks normal to interfacial boundary.

2. FORMULATION OF THE BOUNDARY ELEMENT METHOD

In two-dimensional elasticity, the integral equation for displacements at an interior point, s of a plane elastic region, R with boundary Γ can be derived from Betti's theorem and the solution to Kelvin's problem of the point load in an infinite plane. This integral equation is Somigliana's identity.[8]

$$\mathbf{u}_i(s)=\int_\Gamma U_{ij}(s,Q)\mathbf{P}_j(Q)d\Gamma(Q)-\int_\Gamma P_{ij}(s,Q)\mathbf{u}_j(Q)d\Gamma(Q) \tag{1}$$

where $u_i(s)$ is the displacement vector at interior point, s : $u_j(Q)$, $p_j(Q)$ are the boundary displacements and tractions, and $U_{ij}(s,Q)$, $P_{ij}(s,Q)$ are the displacements and tractions, respectively, in the x_i direction at boundary point Q due to orthogonal unit loads in the x_j direction at s.[9]

Taking $s\rightarrow S(S\in\Gamma)$, a boundary point by a limiting process results in the boundary integral equation, which is an integral constant relating boundary tractions to boundary displacements :

$$C_{ij}(S)\mathbf{u}_j(S)+\int_\Gamma P_{ij}(S,Q)\mathbf{u}_j(Q)d\Gamma(Q)=\int_\Gamma U_{ij}(S,Q)\mathbf{P}_j(Q)d\Gamma(Q) \tag{2}$$

Numerical solutions to the boundary integral equations are found to discretizing over the elements in terms of suitable algebraic functions involving values at certain nodal pointsy associated with the elements. The isoparametric boundary element represents the geometry, displacement and tractions as polynomials :

$$x_i(\eta)=A+B\eta+C\eta^2 \qquad x_i(\eta)=N_l(\eta)x_i^l$$
$$\begin{matrix}\mathbf{u}_i(\eta)\\ \mathbf{p}_i(\eta)\end{matrix}=A'+B'\eta+C'\eta^2 \qquad \begin{matrix}\mathbf{u}_i(\eta)=N_l(\eta)\mathbf{u}_i^l\\ \mathbf{p}_i(\eta)=N_l(\eta)\mathbf{p}_i^l\end{matrix} \tag{3}$$

Where $N_\ell(\eta)$, $\ell=1,2,3$, are quadratic shape functions of the intrinsic coordinate given by

$$N_1(\eta)=\frac{1}{2}\eta(\eta-1)$$
$$N_2(\eta)=\frac{1}{2}\eta(\eta+1)$$
$$N_3(\eta)=(1+\eta)(1-\eta) \tag{4}$$

If the boundary of a particular region is represented by m elements and total 2m nodes, Eq. (2) then becomes

$$C_{ij}(S^a)u_j(S^a)+\sum_{e=1}^{m}u_j(S^{d(e,\ell)})\int_{\Gamma e}P_{ij}(S^a,Q(\eta))N_\ell(\eta)J(\eta)d\eta$$
$$=\sum_{e=1}^{m}p_j(S^{d(e,\ell)})\int_{\Gamma e}U_{ij}(S^a,Q(\eta))N_\ell(\eta)J(\eta)d\eta \tag{5}$$

where S^a is the ath node and $d(e,\ell)$ is the number of the ℓth node of the eth element. Also, $J(\eta)$ is the boundary Jacobian, i.e.,

$$J_i(\eta)=dx_i/d\eta=(dNe(\eta)/d\eta \quad (l=1,2) \tag{6}$$

3. DETERMINATION OF THE STRESS INTENSITY FACTOR

Modification needs to be made in the element modeling to account for crack tip or other singularities. We are going to use special nodes, while essentially retaining standard basic function (Eq. 3). Specifically, for quadratic elements, the internal node is moved to the quarter-point position, as proposed by Henshell[10] and Barsoum,[11] so that, for a crack tip element of length L,

$$r=L\eta^2 \tag{7}$$

u and p are quadratics in $\sqrt{r}$ and have desirable $\sqrt{r}$ singularities. Moreover, the traction may be given its correct $1/\sqrt{r}$ singularity on elements within the material by using the traction singular element.[12]

$$p=(A'+B'\eta+C'\eta^2)/\eta \tag{8}$$

Local displacements in the vicinity of the crack tip are used to determine the stress intensity factors. For points along a face of a crack in two-dimensional region, the local solutions for displacements normal to and tangential to the crack face are

$$u=\frac{K_I}{2G}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(k-1+2\sin^2\frac{\theta}{2}\right)$$
$$v=\frac{K_I}{2G}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(k+1-2\cos^2\frac{\theta}{2}\right) \tag{9}$$

where r is the small distance from the crack tip, G is shearing modulus of elasticity and ν is Poisson's ratio. Here K_I and K_{II} are the mode I and mode II stress intensity factors, and $\kappa=3-4\nu$ for plane strain and $(3-\nu)/(1+\nu)$ for plane stress. By taking $\theta=\pi$ and r small in Eq. (9), we deduce that on the crack surface adjacent to the crack tip, (Fig. 1)

$$K_I=\frac{2G}{k+1}\sqrt{\frac{2\pi}{L}}(4v_3-v_2)$$
$$K_{II}=\frac{2G}{k+1}\sqrt{\frac{2\pi}{L}}(4u_3-u_2) \tag{10}$$

where the points 2 and 3 are defined in Fig. 1

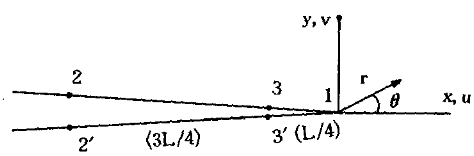

Fig. 1 Quarter-point crack tip element

We also adopt the simple analytic method to obtain the stress intensity factors in bimaterial plate. In stregth of materials, we can analyze the stress of composite materials by using a transformed section corresponding to an equivalent plate made of second material. We extend this method to problems with cracks.

Consider a plate to be loaded in such a way that the top and bottom of the plate are subjected to a uniform displacement. (Fig. 3) Since the moduli of elasticity E_1 and E_2 of two materials in bimaterial plate are different, the expressions obtained for normal stress in each material will be

$$\sigma_1 = E_1 \varepsilon$$
$$\sigma_2 = E_2 \varepsilon$$
$$\varepsilon = \sigma_1/\varepsilon_1 = \sigma_2/E_2$$
$$\sigma_2 = (E_2/E_1)\sigma_1 \quad (11)$$

where σ_1 and σ_2 are the stresses to be acted in the first and second materials in bimaterial plate.

The forces P_1 and P_2 exerted on the first and second materials are illustrated as follows.

$$P = P_1 + P_2$$
$$= (2wt)\sigma_1 + (2bt)(E_2/E_1)\sigma_1 \quad (12)$$

We note that the same force P_2 would be exerted on an element of area E_2/E_1 times of the first material. (Fig. 2)

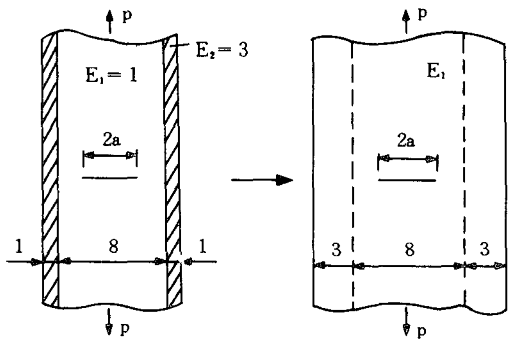

Fig. 2 Transformed section corresponding to an equivalent plate

4. NUMERICAL RESULTS

Numerical results were obtained for the problems of a crack in a two dimensional plate. with the crack running perpendicular to a bimaterial interface. The crack is of length 2a and the boundary conditions are a uniform vertical displacement of the top and bottom of the plate. The problems were analyzed using quarter-point element with traction singularity for two elements adjoining the crack tip and quadratic elements elsewhere. Crack tip element to crack length ratio, L/a is 0.1. The stress intensity factors are calculated using the displacement components on two pairs of points (e.g. 2, 3 in Fig. 1) on the crack tip.

The first case analyzed was a sandwiched layer containing a center crack normal to the interface as shown in Fig. 3. This problem is doubly symmetric and only one-quarter of the plate is discretized. The boundary element mesh is also shown in Fig. 3. The number of bound-

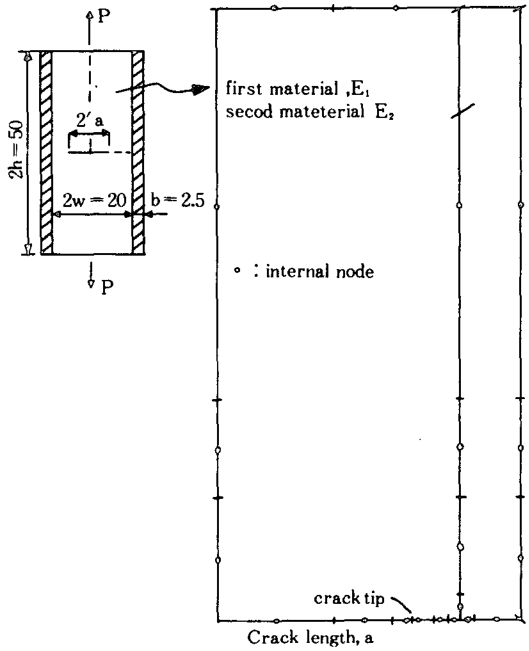

Fig. 3 Boundary element discretization for center crack in bimaterial plate (Model A)

ary elements is 16 for first material containing a crack and 10 for second material. The normalized stress intensity factors from equation (10) are plotted against the ratio a/w for various E_2/E_1 in Fig. 4. The plotted results are compared w하 the pertubation results of Isida[13]. It is clear from Fig. 4 that the results obtained using the boundary element method are quite accurate (less than 3%) for all a/W values up to 0.8.

The second case analyzed was the problems

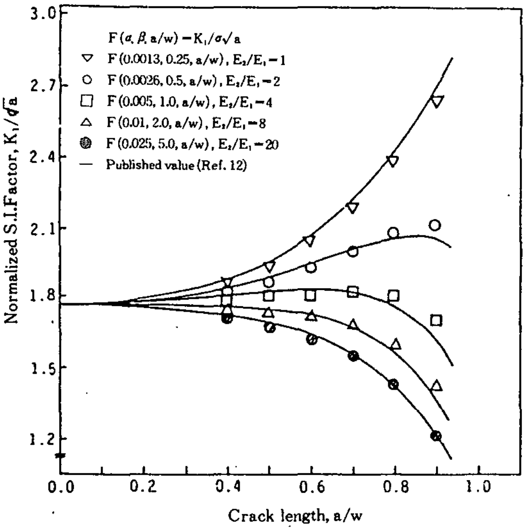

Fig. 4 Normalized K_I vs. crack length for center crack in bimaterial plate (Model A)

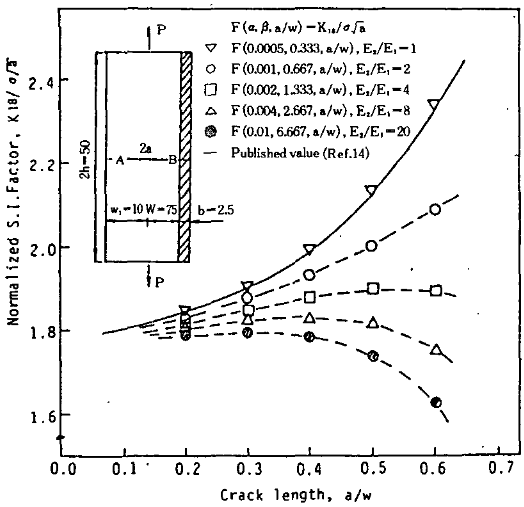

Fig. 5 Normalized K_{IB} vs. crack length for center crack in bimaterial plate (Model B)

of the center and edge cracks normal to a material interface in bimaterial plate as shown in Figs. 5 and 6. The problems are singly symmetric about the crack axis and only one-half of the plate is discretized. The number of boundary elements is 22 for first material containing a crack and 10 for second material. The normalized stress intensity factors at the

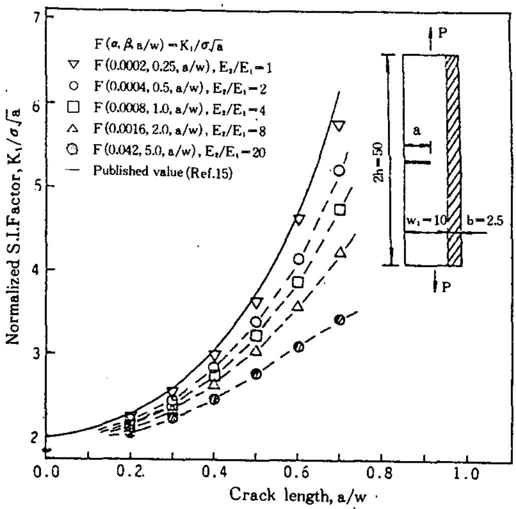

Fig. 6 Normalized K_I vs. crack length for center crack in bimaterial plate (Model C)

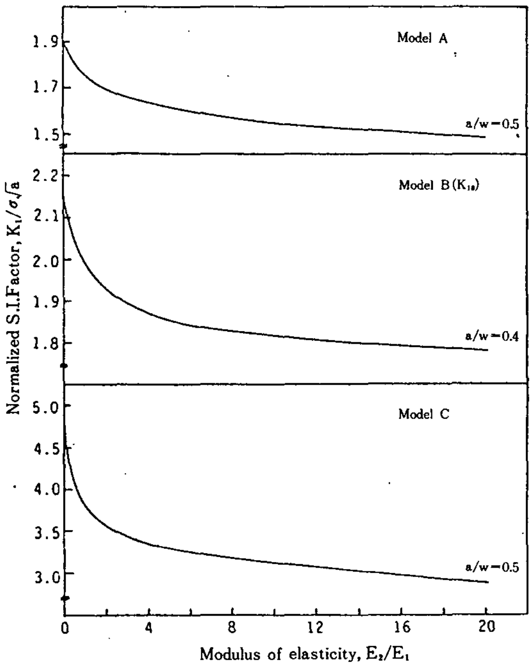

Fig. 7 Normalized K_I vs. modulus of elasticity for each model

nearest crack tip, B from the interface are illustrated in Figs. 5 and 6. The dashed lines given in Figs. 5 and 6 are the collocation results of Idida[14] and Gross el al.[15] respectively for $E_2/E_1=1$.

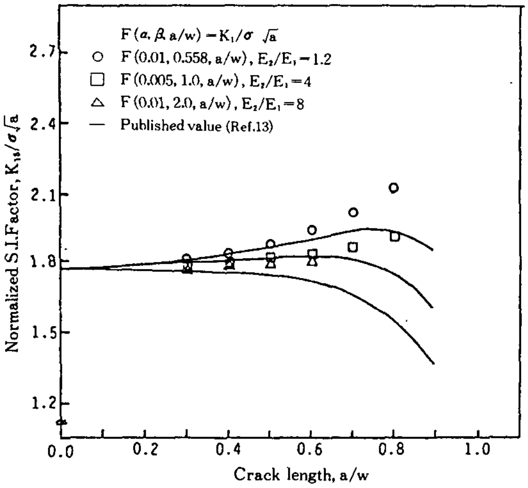

Fig. 8 Normalized K_I vs. crack length for model A by the simple analytic method

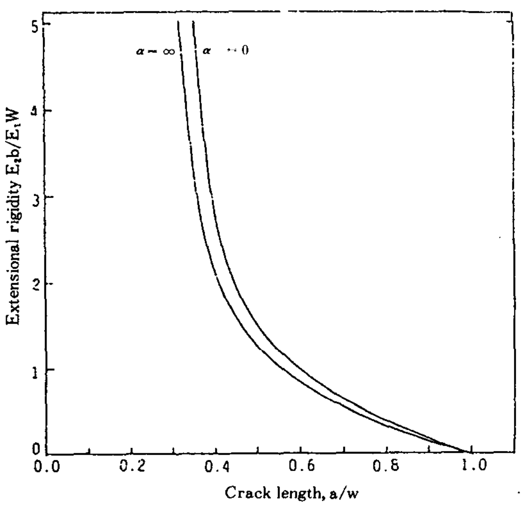

Fig. 9 Extensional rigidity vs. crack length diagram for an analysis of K_I with 3% errors by the simple analytic method (Model A)

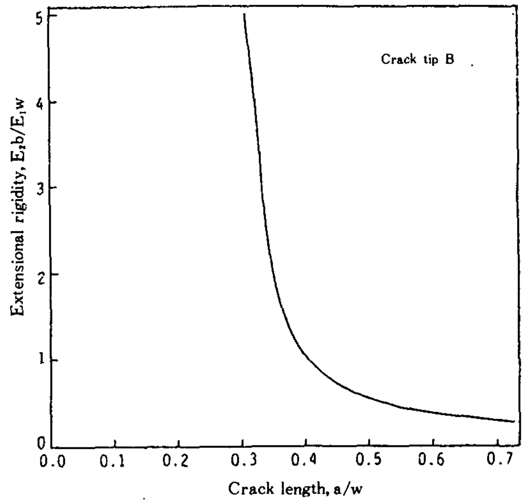

Fig. 10 Extensional rigidity vs. crack length diagram for an analysis of K_I with 3% errors by the simple analytic method (Model B)

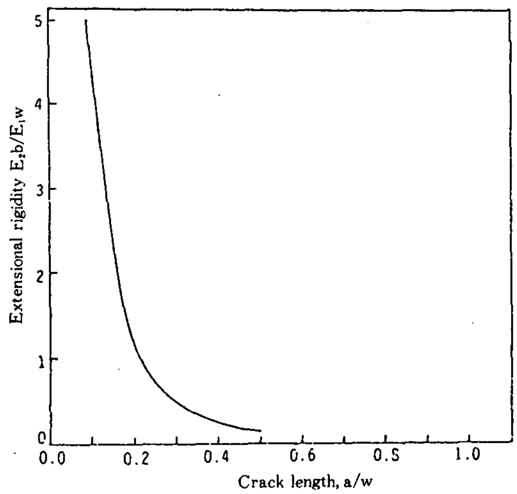

Fig. 11 Extensional rigidity vs. crack length diagram for an analysis of K_I with 3% errors by the simple analytic method (Model C)

Fig. 7 shows the variations of the stress intensity factors with Young's modulus ratio E_2/E_1 for each case. As Fig. 7 shows, the addition of a stiffer second material would cause a drop in the stress intensity factor. The stress intensity factor decreases very fast until E_2/E_1 gets lower than about 4.

The simple analytic method is applied to each case. In Fig. 8 are graphs of the stress intensity factors obtained by the analytic method for the first case. The plotted results are also compared with the perturbation results of Isida.[13] Fig. 8 shows that the accuracy of simple analytic results has been decreased as the values of a/w are increased. We therefore described the applicable range of this method with the values of a/w in Fig. 9. The plotted

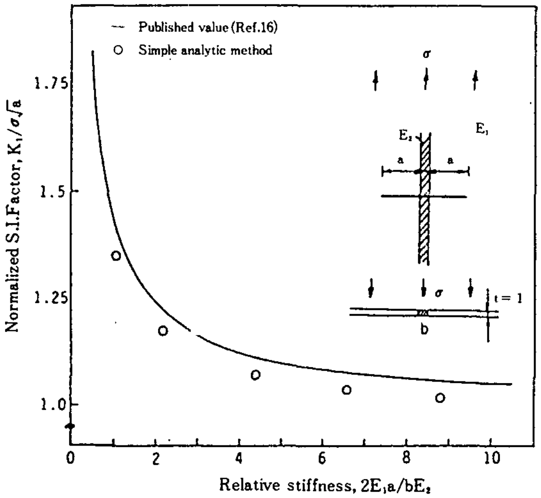

Fig. 12 Normalized K_I for a crack across a second material in a plate subjected to a uniaxial tensile stress by the simple analytic method

results have a discrepancy at most 3% with Isida's and boundary element ones. Fig. 9 shows that the applicable range of the simple analytic method to determine the stress intensity factor for a crack in bimaterial plate mainly depends on extensional rigidity, β. The change caused by varying inertia parameter, α is insignificant. The results for the second case are also given in Figs. 10 and 11.

Moreover, this method can be applied to a crack across the second material in a sheet. The stress intensity factors are plotted against the extensional rigidity in Fig. 12. The dashed line is the theoretical result of Sander and Grief.[16] These results have 5% discrepancies with a theoretical result.

5. CONCLUSIONS

(1) The boundary element method can solve a crack in bimaterial plate to engineering accuracy. Isoparametric quadratic element with the displacement and traction singularities was found to enhance the accuracy of the stress intensity factor with only few boundary elements.

(2) The simple analytic method based on the transformed section corresponding to an equivalent plate made of second material is very effective method for determining the stress intensity factor at crack in bimaterial plate, but which involves somewhat limitation for engineering application.

(3) If a sheet containing a crack is adjoined to a second material having a higher elastic modulus, the stress intensity factor decreases markedly until Young's modulus ratio, E_2/E_1 is about 4.

REFERENCES

1) Sih, G. C. and Chen, E. P., Cracks in Composite Materials, Mechanics of Fracture VI, Martinus Nijhoff, 1981
2) Jaswon, M. A. and Symm, G. T., "Integral Equation Methods in Potential Theory and Elastostatics", Academic press, 1977
3) Altiero, N. J. and Gavaza, S. D., "An Effective Boundary-Integral Approach for the Mixed Boundary Value Problems of Linear Elastostatics", Appl. Math. Modeling, Vol. 3, pp. 99-104, 1979
4) Cruse, T. A., "Numerical Solutions in Three-Dimensional Elastostatics", Int. J. Solids Structures, Vol. 5, pp. 1259-1274, 1969
5) Lachat, J. C. and Watson, J. O., "Effective Numerical Treatment of Boundary Integral Equations : A Formulation for Three-Dimensional Elastostatics," Int. J. Num. Meth. Engng., Vol. 10, pp. 991-1005, 1976
6) Kim, Sang-Chul and Lim, Won-Kyun, "A Prediction of Crack Growth Path by Boundary Element Method", J. of the Society of Naval Architects of Korea, Vol. 25, pp. 39-46, 1988
7) Blandford, G. E., Ingraffea, A. R. and Liggett, J. A., "Two Dimensional Stress

Intensity Factor Computation Using the Boundary Element Method", Int. J. Num. Meth. Engng., Vol. 17, pp. 386-404, 1981

8) Love, A. E. H., A Treatise on the Mathematical Theory of Elasticity, New York, 1944

9) Brebbia, C. A., The Boundary Element Method for Engineering, Pentech press, 1980

10) Henshell, R. D. and Shaw, K. G., "Crack Tip Elements are Unnecessary", Int. J. Num. Meth. Engng., Vol. 9, pp. 495-509, 1975

11) Barsoum, R. S., "On the Use of Isoparametric Finite in Linear Fracture Mechanics", Int. J. Num. Meth. Engng., Vol. 10, pp. 25-37, 1976

12) Cruse, T. A. and Wilson, R. B., "Boundary Integral Equation Method for Elastic Fracture Mechanics", AFOSR-TR-78-0335, pp. 10-11, 1977

13) Isida, M, "Analysis of Stress Intensity Factor for the Tension of a centrally cracked Strip with Stiffened Edges", Engng. Frac. Mech., Vol. 5, pp. 647-665, 1973

14) Isida, M., "Effect of Width and Length on Stress Intensity Factors of internally Cracked Plates under Various Boundary Conditions", Int. J. Frac. Mech., Vol. 7, pp. 301-316, 1971

15) Gross, B, Srawley, J. E. and Brown, W. F., "Stress Intensity Factors for a single Edge Notch Tension Specimen by Boundary Collocation of Stress Function," NASA TND -2395, pp. 1-11, 1964

16) Grief, R. and Sanders, J. L., "The Effect of a Stringer on the Stress in a Cracked Sheet", J. Appl. Mech., Trans. ASME, Vol. 32, pp. 59-65, 1965