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Reliability of a Consecutive-k-out-of-n:G System with Common-Cause Outages

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Abstract

This paper shows the model of a consecutive-k-out-of-n:G system with common-cause outages. The objective is to analytically derive the mean operating time between failures for a non-repairable component system. The average failure time of a system and the system availability are also considered. Then, the model is extended to a system with repairable components and unrestricted repair, in which service times are exponentially distributed.

1. Introduction

Consider a consecutive-k-out-of-n:G system with common−cause outages such that at least k consecutive components should be good for a system to be successfully operating. A common-cause outage is an event having a single external cause with multiple component failures, in which the outage of either line in a power system is independent of the other components. For example, consider the outage of multiple transmission lines in the same tower or on the same right-of-way, due to a single cause such as a flood, tornado or lightening, vehicle or aircraft crash. The objective is to find the mean operating time of a given transmission system. Then, assume that the system does not stop as far as at least k consecutive lines are working. The failure patterns of components are generated independently. Common-cause outage events are not correlated, but a certain component can be failed by more than one external cause. In other words, the sets of failure components caused by external events are not necessarily mutually disjoint.

Alsammarae[1] considers the system availability for the common-cause outage model of 2-state(on or off) 2 lines. In his case, there is a single common-cause outage event. Here, we

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extend to the common-cause outage model with p types of external events where the system is operable if at least k consecutive components are working. Chung [3, 4] and Dhillon [5] present the reliability for the common-cause outage model where 2 or more three-state components are present on the non-repairable system during the system operating period. Mosleh [6] gives the framework for identification, modeling, and quantification of common-cause failures. However, this paper assumes that p types of common-mode outages are identified with the respective component failure sets. Parry [8] presents the possible limitations of the quantitative common-cause failure analysis such as the quality and quantity of data and the lack of guidance for the interpretation of event descriptions.

When the type i external event causing the common-cause forced outages happens, it brings the simultaneous failures of all the components contained in a set S_i. Then, we suppose that the S_i are not necessarily disjoint, i.e., there exist some external events E_i, E_j such that $S_i \cap S_j \neq \phi$ for $i \neq j$. We also assume that the external events E_i causing the forced outages for all components of S_i are occurring in a Poisson stream with mean rate λ_i for $i=1, 2, \cdots, p$, given that there are p types of external events.

This paper first considers a system whose structure is unrepairable during the system operating time, but failed components will wait until the system turns down with less than k consecutive working components, and then be replaced by new ones. The consecutive-k-out-of-n:G system restarts to operate as soon as there exist k consecutive operable components by replacement. That is, the system becomes to a working state with more than $(k-1)$ consecutive workable components after some failure time of a system, and consequently the system operation follows an alternating renewal process. It is also assumed that there is unlimited repair, which is named an ample-server model. Thus, there is no waiting time in the queue with failed components which joined a queue to be serviced earlier. Then, the objective is to find the mean operating time between failures in a steady state for a consecutive-k-out-of-n:G system with common-cause outages.

The model will be extended to a system with repairable components so that each failed component does not wait to be repaired until the system stops. In this case, let us assume that the service times of components in S_i are independent and exponential random variables with mean value $1/\mu_i$ for type i external event. Then, we will find the system availability and the mean failure time of a consecutive-k-out-of-n:G system in a steady state. From the property of the alternating renewal process, the mean operating time between failures can be found directly.

Notation

n number of components in the system

k	minimum number of consecutive operable components required for a system to be successfully operating
u_i	component i
U	$\{u_1, u_2, \cdots, u_n\}$; set of all components
E_i	type i external event
A	$\{E_1, E_2, \cdots, E_p\}$; set of p types of external events
S_i	set of components simultaneously outaged by the event E_i, which is the subset of U for $i=1, 2, \cdots, p$
$\|S_i\|$	number of components contained in the set S_i for $i=1, 2, \cdots, p$
S	$\{S_i : i=1, 2, \cdots, p\}$; set of all S_i
λ_i	mean arrival rate of an external event E_i for $i=1, 2, \cdots, p$
λ	$\sum_{i=1}^{p} \lambda_i$; sum of all the arrival rates of external events
μ	mean service rate of a server to replace each component with a new one
μ_i	mean service rate of a server to repair all the failure components of S_i
Ava_j	availability of system $j(j=R, NR)$
$E(X)_j$	mean operating time between failures for system $j(j=R, NR)$
$E(Y)_j$	mean failure time of system $j(j=R, NR)$
F_j	set of possible numbers of failed components in which system j is down, i.e., a subset of $\{n-k+1, n-k+2, \cdots, n\}$ $(j=R, NR)$
R	system composed of repairable components
NR	system composed of non-repairable components

2. System with Non-Repairable Components

Let us consider the failure pattern that components in a system fail independently according to common-cause outages in which a single external cause results in multiple component failures. The sets $S_i(i=1, 2, \cdots, p)$ of failed components caused by the different external events E_i are not mutually disjoint. That is, there are two external events E_i, $E_j \in A$ such that $S_i \cap S_j \neq \phi$ where each S_m is a nonempty set and $\bigcup_{m=1}^{p} S_m = U$.

System reliability with common-cause outages has been considered in [3−8] and [10], etc. This paper concerns a consecutive-k-out-of-n:G system with common-cause outages and infinite repairmen. We have assumed that the interarrival times of the external events E_i are exponentially distributed with mean rate λ_i, and then all components in a set S_i fail simultaneously, resulting from an external event E_i. Since all the components of the system are assumed to be not repairable in this section, when there are either no k consecutive working components or more than (n-k) failed ones, the system becomes down and stays in the failure (or idle) period until there exist at least k consecutive operable ones. Therefore, when the system is in a working state, the possible numbers of all the consecutive operable components belong to a set {k, k+1,⋯,n}.

At the moment that the system failure period starts, the number of consecutive operable components is in a set {1, 2, ⋯, k−1}, rather than just (k−1). The reason is that the components are subject to common−cause outages causing multiple component failures. For example, a system is in a working condition with k consecutive operable components, but it is down resulting from another external event E_ℓ which causes $m(\geq 2)$ component failures. In this case, there are (k−m) consecutive operable components at the moment that a system fails, which is less than (k−1).

Also, let it be assumed that none of failed components is replaced until the system turns down. The service times to replace by new components are independent and identically distributed with mean rate μ. The system restarts with at least k consecutive operable components. However, where the number of servers is unlimited, the expectation of total system failure time is $1/\mu$, and all components in a system become operable after the system failure time. Then, by the property of an alternating renewal process, the system availability of a consecutive-k-out-of-n:G system in a steady state, Ava_j, can be defined by : (see [9])

$$Ava_j = \lim_{t\to\infty} \Pr\{\text{a system is operating at time } t\}$$

$$= \frac{E(X)_j}{E(X)_j + E(Y)_j} \qquad \text{for } j = R, NR. \tag{1}$$

Since the external events $E_i(i=1, 2, \cdots, p)$ occur in a Poisson process, the instantaneous changes in the number of external events during the operating time are ±1. This system breaks down as soon as the number of consecutive operable components is less than k. However, it is possible to exist less than(k−1) consecutive operable components at the starting moment of the system failure period. This happens because common−cause outages can result in multiple component failures in each set S_i, instead of a single component. We need to choose some types of external events such that the total number of failed components caused by those selected events is greater than or equal to (n−k+1).

The system could be down even by the first external event which causes the simultaneous failures of more than (n−k) components for k > 1. By using the conditional expectation, the mean operating time between failures for a non−repairable system is :

$$E(X)_{NR} = E(E(X|C)) = \sum_{C_i} E(X|c_i) \cdot Pr(c_i)$$

where C is given.

If E_i is the external event of occurring the type i common−cause outage whose number of components, $|S_i|$, is greater than or equal to $(n-k+1)$, i.e., $|S_i| \geq n-k+1$, then $E(X)_{NR}$ becomes :

$$\begin{aligned} &E(X)_{NR} \\ &= \sum_{E \in A} E(X| \ |S_i| \geq n-k+1) \cdot Pr(|S_i| \geq n-k+1) \\ &+ \sum_{\ell=k}^{n-1} [\sum_{E \in A} E(X| \ |S_i| = n-\ell) \cdot Pr(|S_i| = n-\ell) \cdot I_{E_i}(B_\ell) \\ &+ \sum_{\substack{E \in A \\ j \neq i}} E(X| \ |S_j| < n-\ell+1) \cdot Pr(|S_j| < n-\ell+1) \cdot I_{E_j}(B^c_\ell)] \end{aligned} \tag{2}$$

where

$$I_x(B_j) = \begin{cases} 1 & \text{if } x \in B_j \\ 0 & \text{otherwise,} \end{cases}$$

B_ℓ : the set of external events causing the failures of $(n-\ell)$ components, but failing to have the k consecutive operable components,

B^c_ℓ : the set of external events causing the failures of $(n-\ell)$ components, and having the k consecutive operable components,

$Pr(|S_i| \geq n-k+1)$: the probability of occurring the type i common−cause outage event whose number of components failed by event E_i, $|S_i|$, is greater than or equal to $(n-k+1)$,

$Pr(|S_i| < n-k+1)$: the probability of occurring the type i common−cause outage event whose number of components failed by event E_i, $|S_i|$, is less than $(n-k+1)$,

$E(X| \ |S_i| \geq n-k+1)$: the conditional expectation of the operating time, X, given that $|S_i| \geq (n-k+1)$.

Hence, for each $E_i \in A$,

$$Pr(|S_i| \geq n-k+1) = \frac{\lambda_i}{\lambda} \cdot \prod_{E \in A \backslash E_i} (1-\frac{\lambda_j}{\lambda}) \cdot I(|S_i|)$$

where

$A \backslash \{E_i\}$: the set consisting of all the elements of A but E_i, and

$$I(|S_i|) = \begin{cases} 1 & \text{if } |S_i| \geq n-k+1 \\ 0 & \text{otherwise.} \end{cases}$$

The arrival rate of events at the first state is the sum of the arrival rates of all the external events, λ. Then, the conditional expectation of the operating time X given that a single common−cause outage event E_i has occurred with a set S_i of failed components in which $|S_i|$ is at least $(n-k+1)$, $E(X|\ |S_i| \geq n-k+1)$, is :

$$E(X|\ |S_i| \geq n-k+1) = \frac{1}{\lambda}.$$

On the contrary, the system is still working with failed components caused by a single external event E_j such that $|S_j|$ is less than $(n-k+1)$ if k consecutive operable components exist. If another common-cause outage event E_m arrives at the system, then the set of failed components consists of all the elements in S_m but the common element of S_j and S_m, i.e., $S_m \backslash (S_j \cap S_m)$, rather than S_m, since some components in S_m which also belong to S_j have already failed by the earlier common-cause outage event E_j. For the two external events, E_j and E_m, which cause the system failure, we know that $E(X|\ |S_j| < n-\ell+1)$ in Eq. (2) can be described as :

$$\begin{aligned} & E(X|\ |S_j| < n-\ell+1) \\ & = \sum_{\substack{E_m \in A \\ j \neq m}} E(X|\ |S_j| < n-\ell+1,\ |S_j| + |S_m| - |S_j \cap S_m| \geq n-k+1) \\ & \quad \cdot \Pr(|S_j| + |S_m| - |S_j \cap S_m| \geq n-k+1 \mid |S_j| < n-\ell+1) \\ & \quad + \sum_{r=k}^{n-1} \Big[\sum_{\substack{E_m \in A \\ m \neq j}} [E(X|\ |S_j| < n-\ell+1,\ |S_j| + |S_m| - |S_j \cap S_m| = n-r) \\ & \quad \cdot \Pr(|S_j| + |S_m| - |S_j \cap S_m| = n-r \mid |S_j| < n-\ell+1) \cdot I_{E_{jm}}(B_r) \\ & \quad + \sum_{\substack{E_q \in A \\ q \neq m}} E(X|\ |S_j| < n-\ell+1,\ |S_j| + |S_q| - |S_j \cap S_q| < n-r+1) \\ & \quad \cdot \Pr(|S_j| + |S_q| - |S_j \cap S_q| < n-r+1 \mid |S_j| < n-\ell+1) \cdot I_{E_{jq}}(B_r^c) \Big] \end{aligned} \tag{3}$$

where E_{jm} is the union of two external events, E_j and E_m, and $\Pr(|S_j| + |S_m| - |S_j \cap S_m| \geq n-k+1 \mid |S_j| < n-k+1)$ is the conditional probability that the system becomes down by the next event, E_m, given that it is still on a working condition with less than $(n-k+1)$ failed ones resulting from the first external event E_j.

To satisfy both conditions $|S_j| + |S_m| - |S_j \cap S_m| < n-k+1$ and $|S_j| < n-k+1$, it is enough to satisfy that $|S_j| + |S_m| - |S_j \cap S_m| < n-k+1$. Substituting Eq. (3) into Eq. (2), we get :

$$\begin{aligned} & E(X)_{NR} \\ & = \sum_{E_i \in A} E(X|\ |S_i| \geq n-k+1) \cdot \Pr(|S_i| \geq n-k+1) \end{aligned}$$

$$+\sum_{\ell=k}^{n-1}[\sum_{E_i\in A} E(X|\ |S_i|=n-\ell)\cdot Pr(|S_i|=n-\ell)\cdot I_{E_i}(B_\ell)$$

$$+\sum_{\substack{E_j,E_m\in A\\ j\neq m}} E(X|\ |S_j|<n-\ell+1,\ |S_j|+|S_m|-|S_j\cap S_m|\geq n-k+1) \tag{4}$$

$$\cdot Pr(|S_j|<n-\ell+1,\ |S_j|+|S_m|-|S_j\cap S_m|\geq n-k+1)\cdot I_{E_j}(B^c_\ell)$$

$$+\sum_{r=k}^{n-1}\{\sum_{E_m\in A} E(X|\ |S_j|+|S_m|-|S_j\cap S_m|=n-r)$$

$$\cdot Pr(|S_j|+|S_m|-|S_j\cap S_m|=n-r)\cdot I_{E_j}(B^c_\ell)\cdot I_{E_{jm}}(B_r)$$

$$+\sum_{E_q\in A} E(X|\ |S_j|+|S_q|-|S_j\cap S_q|<n-r+1)$$

$$\cdot Pr(|S_j|+|S_q|-|S_j\cap S_q|<n-r+1)\cdot I_{E_j}(B^c_\ell)\cdot I_{E_{jq}}(B^c_r)\}]$$

Eq. (4) can then be generalized to p different external events in the same way. Let C^i represent the set of all the combinations of i external events. For each element C^{ij} of C^i, there are i! permutations of i external events. Given that $Pr(A_{p,j}<n-k+1)=0$, for all the sequences P^{ijm}, $E(X|C)\cdot Pr(C)$ can be shown as :

$$E(X)_{NR}=\sum_{i=1}^{p} E(X|C^i)\cdot Pr(C^i)$$

$$=\sum_{i=1}^{p}\sum_{\substack{C_{ij}\in C^i\\ j=1,2,\cdots,{}_pC_i}}\prod_{q=1}^{i-1}\sum_{\ell_q=k}^{n-1}\sum_{\substack{p^{ijm}\in p^{ij}\\ 1\leq m\leq i!}}\Sigma[E(X|A_{i-1,j}<n-\ell_q+1,\ A_{i,j}\geq n-k+1) \tag{5}$$

$$\cdot Pr(A_{i-1,j}<n-\ell_q+1,\ A_{i,j}\geq n-k+1)\cdot\prod_{q=1}^{i-1} I_{EA_{i-1,j}}(B^c_{\ell q})$$

$$+E(X|A_{i,j}=n-\ell_q)\cdot Pr(A_{i,j}=n-\ell_q)\cdot\prod_{q=1}^{i-2} I_{EA_{i-1,j}}(B^c_{\ell q})\cdot I_{EA_{ij}}(B_{\ell i-1})$$

where

$Pr(C^i)$=the probability that precisely i external events occur before the system failure,

$C^i=\{C^{ij}: j=1,2,\cdots,{}_pC_i\}$ for $i=1,2,\cdots,p$

C^{ij}=subset of A, which is composed of i out of p different external events

$p^{ij}=\{p^{ijm}: 1\leq m\leq i!\}$; set of all the possible sequences of i different external events contained in C^{ij},

$p^{ijm}=(E_{1'}, E_{2'}, \cdots, E_{i'})$; a sequence of i events in C^{ij},

$A_{i,j}$=number of failed components caused by i external events on P^{ijm},

$A_{i-1,j}$=number of failed components caused by all events on P^{ijm} except the last event, $E_{i'}$.

Then, $A_{i,j}$ and $A_{i-1,j}$ can be formulated

$$A_{i,j} = \sum_{E_\ell \in P^{ijm}} |S_\ell| - \sum_{\substack{\ell < m \\ E_\ell, E_m \in P^{ijm}}} |S_\ell \cap S_m| + \sum_{\substack{\ell < m < q \\ E_\ell, E_m, E_q \in P^{ijm}}} |S_\ell \cap S_m \cap S_q| + \cdots + (-1)^{i+1} |\bigcap_{E_\ell \in P^{ijm}} S_\ell|$$

$$A_{i-1,j} = \sum_{E_\ell \in P^{ijm} \setminus \{E_{i'}\}} |S_\ell| - \sum_{\substack{\ell < m \\ E_\ell, E_m \in P^{ijm} \setminus \{E_{i'}\}}} |S_\ell \cap S_m| + \sum_{\substack{\ell < m < q \\ E_\ell, E_m, E_q \in P^{ijm} \setminus \{E_{i'}\}}} |S_\ell \cap S_m \cap S_q| + \cdots + (-1)^{i} |\bigcap_{E_\ell \in P^{ijm} \setminus \{E_{i'}\}} S_\ell|$$

where $E_{i'}$ is the last event on P^{ijm}.

For any sequence of external events, P^{ijm}, the system operating time up to failure, X, is the sum of the inverses of the arrival rates of the remaining events at the ℓ^{th} state for $\ell=1, 2, \cdots, i-1$. Given $A_{i-1,j} < n-k+1$ and $A_{i,j} \geq n-k+1$, the mean time up to failure is the product of the system operating time and the probability that the events contained in C^{ij} occur according to the sequence P^{ijm} ; that is,

$$E(X | A_{i-1,j} < n-k+1,\ A_{i,j} \geq n-k+1)$$
$$= \left(\sum_{\ell=0}^{i-1} \frac{1}{\lambda - \sum_{j'=1}^{\ell} \lambda_{j'}}\right) \cdot \prod_{\ell=1}^{i} \frac{\lambda_{\ell'}}{\Sigma - \sum_{j'=1}^{\ell-1} \lambda_{j'}} \quad . \tag{6}$$

$Pr(A_{i-1,j} < n-k+1,\ A_{i,j} \geq n-k+1)$ is the probability of occurring i external events on P^{ijm} causing the system failure such that $A_{i-1,j} < n-k+1$ and $A_{i,j} \geq n-k+1$. Therefore, it is the product of the probabilities of occurring all the external events E_ℓ in C^{ij} and of not occurring the other events if the total number of failed components is greater than or equal to $(n-k+1)$ and the number of failed ones caused by those but the last event on P^{ijm} is less than $(n-k+1)$. For each P^{ijm} of the set P^{ij}, it is

$$Pr(A_{i-1,j} < n-k+1,\ A_{i,j} \geq n-k+1)$$
$$= \prod_{E_\ell \in C^{ij}} \frac{\lambda_\ell}{\lambda} \cdot \prod_{E_\ell \in A \setminus C^{ij}} \left(1 - \frac{\lambda_\ell}{\lambda}\right) \cdot I(A_{i-1,j}, A_{i,j}) \tag{7}$$

where

$$I(A_{i-1,j}, A_{i,j}) = \begin{cases} 1 & \text{if } A_{i-1,j} < n-k+1 \text{ and } A_{i,j} \geq n-k+1 \\ 0 & \text{otherwise.} \end{cases}$$

And,

$$Pr(A_{i,j} = n - \ell_q)$$
$$= \prod_{E_\ell \in C^{ij}} \frac{\lambda_\ell}{\lambda} \cdot \prod_{E_\ell \in A \setminus C^{ij}} \left(1 - \frac{\lambda_\ell}{\lambda}\right) \cdot I(A_{i,j})$$

where

$$I(A_{i,j}) = \begin{cases} 1 & \text{if } A_{i,j} = n - \ell_q \\ 0 & \text{otherwise.} \end{cases}$$

Then, substituting Eqs. (6) and (7) into Eq. (5), $E(X)_{NR}$ becomes :

$$
\begin{aligned}
E(X)_{NR} &= \sum_{i=1}^{p} \sum_{\substack{C'' \in C' \\ j=1,2,\cdots,{}_pC_i}} \prod_{E_\ell \in C'} \frac{\lambda_\ell}{\lambda} \cdot \prod_{E_\ell \in A \setminus C''} \left(1-\frac{\lambda_\ell}{\lambda}\right) \Big[\prod_{q=1}^{i-1} \sum_{\ell_*=k}^{n-1} \\
& \sum_{\substack{p''' \in p'' \\ 1 \le m \le i!}} \left(\sum_{\ell=0}^{i-1} \frac{1}{\lambda - \sum_{j=1}^{\ell} \lambda_{j'}} \right) \cdot \prod_{\ell=1}^{i} \frac{\lambda_{\ell'}}{\sum_{E_j \in c'} \lambda_j - \sum_{j'=1}^{\ell-1} \lambda_{j'}} \{ I(A_{i-1,j}, A_{i,j}) \\
& \cdot \prod_{q=1}^{i-1} I_{EA_{i-1,j}}(B_{\ell q}^{c}) + I_{A_{i,j}} \prod_{q=1}^{i-2} I_{EA_{i-1,j}}(B_{\ell q}^{c}) \cdot I_{EA_{ij}}(B_{\ell i-1}) \} \Big]
\end{aligned}
\tag{8}
$$

The set of failed components at the starting point of the idle period is P^{ijm}. However, the service times to replace are identically distributed for all the components, and thus the mean failure(idle) time, $E(Y)_{NR}$, becomes $1/\mu$ where μ is the mean service rate in the ample−server model. After the system idle period, all the components become operable, and therefore, this system follows the alternating renewal process. By substituting $E(X)_{NR}$ shown in Eq. (8) and $E(Y)_{NR} = 1/\mu$ into Eq. (1), the system availability of a consecutive-k-out-of-n:G with unrepairable components, Ava_{NR}, can be found directly.

3. System with Repairable Components

In this section, let us consider the case that components are repairable and there is infinite repair in the system. Components failed from common-cause outages will immediately join the queue to be repaired, and return to the system as soon as they are serviced. Now, assume that the sets $S_i(i=1,2,\cdots,p)$ of failed components caused by different external events E_i are mutually exclusive. Then, according to the external events E_i, the set U of all components can be partitioned into mutually disjoint sets $S_1, S_2, \cdots, S_p$ such that $S_i \cap S_j = \phi$ for any i, j, and $\bigcup_{i=1}^{p} S_i = U$ where each S_i is not an empty set.

The interarrival times of external events E_i with mean rate λ_i are independent and exponentially distributed, resulting in the failures of components in S_i. Furthermore, all components in S_i failed by external event E_i are repaired with mean rate μ_i in which repair times are also independent, and exponentially distributed. The system is in a working condition with the number of consecutive operable components in a set $\{k, k+1, \cdots, n\}$. If a system does not have at least k consecutive operable components, it begins the idle period and restarts to operate as soon as there exist at least k consecutive good components by repair.

First, define a subset T_j of S which is composed of some S_i such that the total number of components of S_i contained in T_j is N_j. In other words, N_j is the sum of $|S_i|$ for all $S_i \in T_j$. Since p different external events can happen, there are total 2^p subsets of S. If all the possible N_j are arranged in an increasing order, it is obvious that $N_2 = \text{Min}\{|S_i| : \text{for each } S_i \in S\}$ where T_1 is an empty set with $N_1 = 0$.

Furthermore,

$N_3 = \text{Min}\{|S_j| : \text{for each } S_j \in S \backslash S\{S_i\} \text{ where } N_2 = |S_i|\}$, and,

$$N_1 < N_2 \leq \cdots \leq N_{r-1} < n-k+1 \leq N_r \leq \cdots < N_{2p}$$

where $N_1 = 0$, and $T_{2p} = S$ with $N_{2p} = n$. Therefore, the system is still on an operating condition for the subset T_{r-1} with the number of failed components, N_{r-1}, if there exists k consecutive operable components in the remaining set. However, the system is down with either T_r whose total number of components is N_r, or no k consecutive operable components.

Notice that it is possible to have the same N_j, although the corresponding subsets T_j are not identical. For some T_j with N_j satisfying that $N_j < (n-k+1)$, define a complementary set of T_j, $S \backslash T_j$, where $S = \{S_i : \text{for all } i=1, 2, \cdots, p\}$ so that each element is $S_{Tj}^c \in S \backslash T_j$. The system is still working with a set T_j in which the number of failure components, N_j, is less than $(n-k+1)$ and there are k consecutive operable components in $S \backslash T_j$. With one more event E_{Tj}^c the total number of failed components is greater than $(n-k)$ or there exist no k consecutive operable components although the number of operable components is greater than or equal to k. Therefore, a system is operational when all components in T_j fail, but stops with another failure event E_{Tj}^c whose total number of failed components is $(N_j + |S_{Tj}^c|)$. The number of failed components which results in the system stop with one more event E_{Tj}^c is an element of a set G_{Tj} defined as follows :

$G_{Tj} = \{N_j + |S_{Tj}^c| : \text{for each } S_{Tj}^c \in S \backslash T_j$ and $N_j < (n-k+1)$ such that either $N_j + |S_{Tj}^c| \geq (n-k+1)$ or there exist no k consecutive operable components in the remaining set$\}$.

And,

$G_T = \{G_{Tj}$: for all j such that a system is working with a subset T_j, but down with failed components in $T_j \cup \{S_{Tj}^c\}\}$.

Then, the probability of a system operating with i unfailed components and k consecutive operational components, R_i, can be written as follows :

$$R_i = \frac{P(i)}{\sum_{i \in G} P(j)} \qquad \text{for } i \in G_1$$

where

$P(i)$: probability of a system having i operable components

$G = G_1 \cup G_2$: set of numbers of operable components for all the possible states

$G_1 = \{n-N_j : j=1, 2, \cdots, r-1\}$: set of possible numbers of operable components before a system fails such that N_j is less than $(n-k+1)$ and there are k consecutive operable components

$G_2 = \{n-m \mid \text{for all } m \in G_{rj} \in G_r\}$: set of possible numbers of operable components at the moment that a system fails.

Using R_i, the system availability in a steady state Ava_R can be shown such as : (see [2])

$$Ava_R = \sum_{i \in G_1} R_i = \frac{1}{1 + \dfrac{\sum_{m \in G_2} P(m)}{\sum_{j \in G_1} P(j)}} \qquad (9)$$

where

$$P(n-N_j) = \sum_{T_{ji}^c \in T_j^c} \prod_{S_m \in T_{ji}} \frac{\mu_m}{\lambda_m + \mu_m} \cdot \prod_{S_\ell \in T_{ji}^c} \frac{\lambda_\ell}{\lambda_\ell + \mu_\ell}$$
$$= \frac{1}{\prod_{m=1}^{p}(1+\rho_m)} \cdot \sum_{T_{ji}^c \in T_j^c} \left(\prod_{S_\ell \in T_{ji}^c} \rho_\ell\right)$$

: probability of a system having $(n-N_j)$ operable components

$T_j^c = \{T_{ji}^c \subseteq S$ such that the total number of operable components is $(n-N_j)\}$

$T_{ji}^c = S \backslash T_{ji}$

T_{ji} : subset of S, whose total number of components is $\sum_{S_\ell \in T_{ji}} |S_\ell| = N_j$

$S = \{S_i : i=1, 2, \cdots, p\}$

$\rho_\ell = \dfrac{\lambda_\ell}{\mu_\ell}$ for each ℓ such that $S_\ell \in S$.

Notice that for the unrestricted repair, $1/(1+\rho_\ell)$ and $\rho_\ell/(1+\rho_\ell)$ are the availability and unavailability of components contained in S_ℓ in a steady state, respectively. To find the mean failure(idle) time, by using the conditional expectation, the mean repair rate of more than $(n-k)$ failed components or no k consecutive operable components, $\mu(F_R)$, can be shown such as :

$\mu(F_R)$

= E(sum of all service rates | number of failed components in F_R)

· Pr(number of failed components in F_R)

+ E(sum of all service rates | no k consecutive operable components with less than $(n-k+1)$ failed ones) · Pr(no k consecutive operable components with less than $(n-k+1)$ failed ones)

$$= \sum_{T_i \in T} \sum_{T_{ij} \in T_i} \left(\sum_{S_\ell \in T_{ij}} \mu_\ell\right) \cdot \prod_{S_\ell \in T_{ij}} \frac{\rho_\ell}{1+\rho_\ell} \cdot \prod_{S_m \in T_{ij}^c} \frac{1}{1+\rho_m}$$

where

T = {T_j with N_j number of failed components if either $n-k+1 \le N_j \le n$ or there exist no k consecutive operable components}

$T_j = \{T \subseteq S : \sum_{S_\ell \in T_m} |S_\ell| = N_j\}$.

Then, the mean idle time of a system, E(Y), becomes :

$$E(Y)_R = \frac{\prod_{i=1}^{p}(1+\rho_i)}{\sum_{T_j \in T} \sum_{T_m \in T_j} \prod_{S_\ell \in T_m} \rho_\ell \cdot (\sum_{S_\ell \in T_m} \mu_\ell)} \tag{10}$$

From Eq. (1), $E(X)_R = \frac{Ava_R \cdot E(Y)_R}{1-Ava_R}$, and substituting Eqs. (9) and (10), the mean operating time between failures for a consecutive-k-out-of-n:G system with repairable components, $E(X)_R$, is shown as :

$$E(X)_R = \frac{\prod_{i=1}^{p}(1+\rho_i) \cdot \sum_{j \in G_1} P(j)}{\sum_{T_j \in T} \sum_{T_m \in T_j} \prod_{S_\ell \in T_m} \rho_\ell \cdot (\sum_{S_\ell \in T_m} \mu_\ell) \cdot \{\sum_{j=G_2} P(j)\}}. \tag{11}$$

4. Numerical Example

Consider the electric model that the electricity is supplied through total 6 electric power lines. In order to prevent the overcurrent caused by the line faults, a fuse is installed on each power supply line that is connected to the common bus. The line fault occurs between the power fuse and the common bus, and then the linearly ordered set U of all the power lines is $\{u_1, u_2, \cdots, u_6\}$ and n=6. Here, we assume that at least 4 power lines should be operating consecutively in order to continue the power supply, that is, k=4.

For three external events, E_1-E_3, causing the common-cause outages, assume that the interarrival times are exponentially distributed, where the arrival rates of events E_i for i=1, 2, 3 are 0.5/yr, 0.8/yr, and 0.2/yr, respectively. The common-cause outage event E_1 brings the simultaneous faults of power lines $\{u_1\}$, $u_4\}$, and E_2 $\{u_3, u_4\}$, and E_3 $\{u_4, u_5, u_6\}$. Thus, $A=\{E_1, E_2, E_3\}$. During the operating period of the non-repairable system, there are four feasible sequences to be considered for the occurrence of external events, that is, $\{E_1, E_2\}$, $\{E_1, E_3\}$, $\{E_2\}$, and $\{E_3\}$. Then, by using Eq. (8), the mean operating time between failures for the non-repairable system, $E(X)_{NR}$, is 0.35651 year. In addition, the mean operating time for the repairable system, $E(X)_R$, can be also found from Eq. (11).

5. Conclusions

In this paper, a solution procedure was developed to find the mean operating time between failures for consecutive-k-out-of-n:G non-repairable and repairable component systems with the infinite number of servers in which n components are subject to common-cause outages. In addition, the mean system failure time and the system availability were also derived.

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