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Average Length and Bounds on the Busy Period for a k-out-of-n : G System with Non-identical Components

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Abstract

The model of a k-out-of-n : G repairable system with identical components is extended to a repairable system with n different components. The objective is to analytically derive the mean time of the busy period for a k-out-of-n : G system with unrestricted repair. Then, the lower and upper bounds on the average time of the busy period of the n-component system with restricted repair are also shown.

1. Problem Description

Consider a k-out-of-n : G repairable system with n different components such that at least k of n components should be good for a system to be successfully operating. The failure pattern of components and the service scheme to repair are probabilistic. Service requests of each component i are generated independently in homogeneous(stationary) Poisson streams with mean rate, λ_i. The service time to repair each failed component i is independent, and exponentially distributed with mean rate, μ_i.

As a queue discipline which is the selection procedure of failed components for repair when a queue has formed, this paper first considers an ample-repairman model for which there is an infinite number of repairmen available. In this case, there is no waiting time in the queue and the unavailable time of each component in the system is only its repair time. Secondly, a First-Come-First-Served repair discipline is considered, in which there is a single repairman.

Now, the objective of this paper is to calculate the average lengths of the busy(system operating) period and the idle(system breakdown) period of a k-out-n : G system with unrestricted repair. Then, the upper and lower bounds on the mean times of the busy and the

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idle periods for a single repairman system, i.e., a restricted system will be found.

First, let us define the terms that will be used in the following sections.

Notation

n	number of components in the system
k	minimum number of operable components required for a system to be successfully operating
λ_i	mean failure rate of component i(i=1, 2, ⋯, n)
μ_i	mean repair rate of component i(i=1, 2, ⋯, n)
P(i)	probability of having i operable components in an n-component system
E(UX)	expectation of the busy(system operating) period for an unrestricted repair system, i.e., a system with an infinite number of repairmen
E(RX)	expectation of the busy(system operating) period for a restricted repair system, i.e., a single repairman system
E(UY)	expectation of the idle(system breakdown) period for an unrestricted repair system
E(RY)	expectation of the idle(system breakdown) period for a restricted repair system
$E(\cdot)_{\lambda,\mu}$	expectation of (·) for an n-identical component system in which λ and μ are mean failure and mean repair rates, respectively
UR	availability of an unrestricted repair system in a steady state
RR1(λ,μ)	availability of a restricted repair system with n identical components in a steady state in which λ and μ are mean failure and mean repair rates, respectively
RR2	availability of a restricted repair system with n different components in a steady state.

2. Mean Time of the Busy Period for Unrestricted Repair System

Assuming that n components are identical such that the interfailure and the repair times of each component are exponentially distributed with mean rates, λ and μ, respectively, the average length of the busy period for a k-out-of-n : G system with unrestricted repair is shown in [2]. This formula is also derived in [1], by the different way. Now, we assume that a k-out-of-n : G system has n different components in which component i has the mean failure rate λ_i, and the mean repair rate μ_i. Then, by the property of alternating renewal theory with Poisson processes, the system availability in a steady state, UR, can be written as follows :

$$UR = \lim_{t\to\infty} \Pr\{\text{a system is operating at time } t\} = \frac{E(UX)}{E(UX)+E(UY)} \tag{1}$$

The set of the numbers of operable components is $\{k, k+1, \cdots, n-1, n\}$ when a system is successfully operating. Because a system turns down as soon as $(n-k+1)$ components are failed, that is, when a system has only $(k-1)$ operable components, the system remains idle until there are at least k operable components, by repairing failed components.

Then, the probability of a system operating with i unfailed components, UR_i, becomes :

$$UR_i = \frac{P(i)}{\sum_{j=k-1}^{n} P(j)} \qquad \text{for } i=k, k+1, \cdots, n.$$

By using the probability of a system operating with i operable components, UR_i, the system availability in a steady state, UR, of Eq. (1) can be rewritten as follows (see [11]) :

$$UR = \sum_{i=k}^{n} UR_i = \sum_{i=k}^{n} \frac{P(i)}{\sum_{j=k-1}^{n} P(j)} = \frac{1}{1+\frac{P(k-1)}{\sum_{j=k}^{n} P(j)}} \tag{2}$$

where each $P(j)$ can be calculated by

$$P(j) = \sum_{\substack{S_i^j \in C_j \\ i=1,2,\cdots,{}_nC_j}} \prod_{\ell \in S_i^j} \frac{\mu_\ell}{\lambda_\ell+\mu_\ell} \cdot \prod_{m \in S^n \setminus S_i^j} \left(1-\frac{\mu_m}{\lambda_m+\mu_m}\right)$$

$$= \sum_{\substack{S_i^j \in C_j \\ i=1,2,\cdots,{}_nC_j}} \prod_{\ell \in S_i^j} \frac{1}{1+\rho_\ell} \cdot \prod_{m \in S^n \setminus S_i^j} \left(1-\frac{1}{1+\rho_m}\right)$$

$$= \frac{1}{\prod_{\ell=1}^{n}(1+\rho_\ell)} \cdot \sum_{\substack{S_i^j \in C_j \\ i=1,2,\cdots,{}_nC_j}} \prod_{m \in S^n \setminus S_i^j} \rho_m \tag{3}$$

where

$S^n=\{1, 2, \cdots, n\}$

S_i^j=a set with j different components for $i=1, 2, \cdots, {}_nC_j$ and ${}_nC_j = \frac{n!}{(n-j)!j!}$

$C_j = \{S_1^j, S_2^j, \cdots, S_{{}_nC_j}^j\}$ for $j = k-1, k, \cdots, n-1, n$

$\rho_m = \lambda_m/\mu_m$ for $m = 1, 2, \cdots, n$.

Notice that $\mu_i/(\lambda_i+\mu_i)$ is the availability of component i $(i=1, 2, \cdots, n)$ in a steady state.

As soon as (n−k+1) components are failed, a system with an infinite number of repairmen is down and begins to work after the expected repairtime of (n−k+1) failed components, $E(UY)=1/\mu_{n-k+1}$. Note that μ_{n-k+1} is the expectation of the sum of repair rates for (n−k+1) components in the infinite repairmen system. Using the conditional expectation, μ_{n-k+1} can be shown as :

$$\mu_{n-k+1}=E(\text{sum of all repair rates}\,|\,(n-k+1)\text{ components failed})\cdot \Pr\{(n-k+1)\text{ components failed}\}$$
$$=\sum_{\substack{S_i^{n-k+1}\in C_{n-k+1}\\ i=1,2,\cdots,{}_nC_{n-k+1}}}\ \prod_{j\in S_i^{n-k+1}}\frac{\rho_j}{1+\rho_j}\cdot\prod_{j\in S^n\setminus S_i^{n-k+1}}\left(\frac{1}{1+\rho_j}\right)\cdot\left(\sum_{j\in S_i^{n-k+1}}\mu_j\right).$$

Then, E(UY) can be found as follows :

$$E(UY)=\frac{\sum_{j=1}^{n}(1+\rho_j)}{\sum_{\substack{S_i^{n-k+1}\in C_2\\ i=1,2,\cdots,{}_nC_{n-k+1}}}\ \prod_{j\in S_i^{n-k+1}}\rho_j\cdot\left(\sum_{j\in S_i^{n-k+1}}\mu_j\right)} \tag{4}$$

where

$$\rho_j=\lambda_j/\mu_j.$$

Replacing UR of Eq. (2) into Eq. (1), E(UX) can be written as :

$$E(UX)=\frac{E(UY)\sum_{j=k}^{n}P(j)}{P(k-1)}\ . \tag{5}$$

By substituting Eqs. (3) and (4) into Eq. (5), the mean time of the busy period for a k out-of-n : G system with unrestricted repair, E(UX), is :

$$E(UX)=\frac{\left\{\prod_{j=1}^{n}(1+\rho_j)\right\}\cdot\sum_{j=k}^{n}\left(\sum_{\substack{S_i^j\in C_1\\ i=1,2,\cdots,{}_nC_j}}\ \prod_{m\in S^n\setminus S_i^j}\rho_m\right)}{\left\{\sum_{\substack{S_i^{n-k+1}\in C_2\\ i=1,2,\cdots,{}_nC_{n-k+1}}}\ \prod_{j\in S_i^{n-k+1}}\rho_j\left(\sum_{j\in S_i^{n-k+1}}\mu_j\right)\right\}\cdot\left(\sum_{\substack{S_i^{k-1}\in C_1\\ i=1,2,\cdots,{}_nC_{k-1}}}\ \prod_{m\in S^n\setminus S_i^{k-1}}{}'\rho_m\right)}$$

3. Bounds on the Mean Time of the Busy Period for Restricted Repair System

When the interfailure and repair times of components in a k-out-of-n : G system with restricted repair (a single repairman) are independent, and identically distributed, the average length of the busy period for a system, $E(RX)_{\lambda,\mu}$, is shown in [2] as follows :

$$E(RX)_{\lambda,\mu}=\frac{1}{\lambda}\sum_{i=0}^{n-k}\frac{1}{(i+1)!\ {}_{k+i}C_{i+1}(\lambda/\mu)^{i}} \tag{6}$$

where λ and μ are the mean failure and the mean repair rates of each component, respectively. And, the mean time of the idle period, $E(RY)_{\lambda,\mu}$, is :

$$E(RY)_{\lambda,\mu}=\frac{1}{\mu} \tag{7}$$

Then, by utilizing Eqs. (6) and (7), the system availability, $RR1(\lambda,\mu)$, can be calculated by :

$$RR1(\lambda,\mu)=\frac{1}{1+\dfrac{1}{\sum_{i=1}^{n-k+1}\dfrac{1}{i!\,{}_{k+i-1}C_i\,\rho^{i}}}} \qquad \text{for } \rho=\frac{\lambda}{\mu}.$$

Now, to find the lower and upper bounds on the system availability for the restricted repair system with n different components, let us first define the following :

$$\lambda_{max}=\max\{\lambda_i \text{ for } i=1,2,\cdots n\},\quad \lambda_{min}=\min\{\lambda_i \text{ for } i=1,2,\cdots n\}$$
$$\mu_{max}=\max\{\mu_i \text{ for } i=1,2,\cdots n\},\quad \mu_{min}=\min\{\mu_i \text{ for } i=1,2,\cdots n\}.$$

Also, define the new values, $\hat{\rho}$ and $\tilde{\rho}$, such as :

$$\hat{\rho}=\frac{\lambda_{max}}{\mu_{min}} \quad \text{and} \quad \tilde{\rho}=\frac{\lambda_{min}}{\mu_{max}}.$$

Then, it is obvious that $\lambda_{max}\geq\lambda_i$ and $\mu_{min}\leq\mu_j$ for any i, j. Consider the system with n identical components in which the mean failure and repair rates of each component are λ_{max} and μ_{min}, respectively. During the mean time of the busy period for a non-identical component system, the sum of failure rates for an identical component system with failure rate λ_{max} is greater than that for a non-identical component system where $\lambda_{max}\neq\lambda_i$ for some i. And, the mean service time of an identical component system, $1/\mu_{min}$, satisfies that $1/\mu_{min}\geq\mu_j$ for any j, so the repair rate during $1/\mu_{min}$ is greater than or at least equal to that during any service time $1/\mu_j$.

Therefore, the mean time of the busy period for a k-out-of-n :G identical component system with λ_{max} and μ_{min} satisfies :

$$E(RX)_{\lambda max, \mu min} \leqslant E(RX)$$

so that $E(RX)_{\lambda max, \mu min}$ is the lower bound on the busy period for a k-out-of-n : G system with different components and restricted repair. To find $E(RX)_{\lambda max, \mu min}$, utilizing Eq. (6), it becomes :

$$E(RX)_{\lambda max, \mu min} = \frac{1}{\mu_{min} \cdot \sum_{\ell=1}^{n-k+1} \ell ! \cdot {}_{k+\ell-1}C_{\ell} \cdot \hat{\rho}^{\ell}}$$

where $\hat{\rho} = \lambda_{max} / \mu_{min}$.

Furthermore, it is true that $E(RY)_{\lambda max, \mu min} = 1/\mu_{min}$ and $E(RY)_{\lambda max, \mu min} \geq E(RY)$, and we have the result that

$$\frac{E(RX)_{\lambda max, \mu min}}{E(RY)_{\lambda max, \mu min}} \leqslant \frac{E(RX)}{E(RY)} \Rightarrow RR1_{\lambda max, \mu min} \leqslant RR2.$$

That is, the identical component system availability with λ_{max} and μ_{min} is less than or equal to that of non-identical component system with λ_i and μ_i for i = 1, 2, ⋯, n.

By the similar way to the lower bound, it is obvious that the upper bound of E(RX) is the mean time of the busy period for a k-out-of-n : G identical component system with λ_{min} and μ_{max}, that is, $E(RX) \leq E(RX)_{\lambda min, \mu max}$.

Then, the upper bound of E(RX), $E(RX)_{\lambda mix, \mu max}$, can be shown as :

$$E(RX)_{\lambda min, \mu max} = \frac{1}{\mu_{max} \cdot \sum_{\ell=1}^{n-k+1} \ell ! \cdot {}_{k+\ell-1}C_{\ell} \cdot \tilde{\rho}^{\ell}}$$

where $\tilde{\rho} = \lambda_{min} / \mu_{max}$.

Also, we know that $E(RY)_{\lambda min, \mu max} = 1/\mu_{max}$ and $E(RY)_{\lambda min, \mu max} \leq E(RY)$.

By the same way as above, the identical component system availability with λ_{min} and μ_{max} is greater than or equal to that of non-identical component system. That is, $RR2 \leq RR1(\lambda_{min}, \mu_{max})$.

4. Conclusions

The model of a k-out-of-n : G repairable system with non-identical components is considered. The mean time of the busy period for a k-out-of-n : G System with unrestricted re pair is analytically derived. In addition, the lower and upper bounds on the expectation of the busy period for the restricted repair system are also found.

References

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