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Notes on Reducing Mixed Integer Knapsack Problems

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Abstract

We consider 0-1 mixed integer knapsack problems. They turn out to be no more difficult to solve than the corresponding 0-1 pure integer knapsack problems with efficient pseudopolynomial time algorithms.

1. Introduction

Consider the following two versions of the 0-1 mixed integer knapsack problem.

$$\begin{aligned}(P0)\ Max\ & \sum_{i\in I} a_i x_i + by \\ s.t.\ & \sum_{i\in I} c_i x_i + y \le K \\ & x_i \in \{0,1\},\ \forall i\in I \\ & y \ge 0\end{aligned}$$

$$\begin{aligned}(Q0)\ Min\ & \sum_{j\in J} d_j x_j + ey \\ s.t.\ & \sum_{j\in J} f_j x_j + y \ge L \\ & x_j \in \{0,1\},\ \forall j\in J \\ & y \ge 0\end{aligned}$$

where all coeffcients are positive. Without loss of generality, we assume that the x_i's are arranged in nonincreasing order of a_i/c_i $\forall i \in I$ and the x_j's are assumed to be arranged in nondecreasing order of d_j/f_j $\forall j \in J$. It is trivial to see that the 0-1 mixed integer knapsack problem with multiple continuous variables reduces to one with a single continuous variable.

0-1 mixed integer knapsack problems are easily seen as substructres in a monolithic production/manufacturing problem. A typical example of $(P0)$ is the investment problem where a fraction of a certain investment option is allowed within a budget constraint. As an example of $(Q0)$, we can consider the minimum cost packing problem where a fraction of a certain item is

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allowed to be packed subject to a minimum required container size. It is well known that mixed integer programs are generally harder to solve than pure integer programs.[3] In this paper, we will show that 0-1 mixed integer knapsack problems are not necessarily harder to solve than the corresponding 0-1 pure integer knapsack problems with efficient pseudopolynomial time algorithms.[2]

2. Notation

Given an optimization problem (P), $OV(P)$, $OS(P)$ and $FR(P)$ denote an optimal value, a set of optimal solutions and feasible region of (P) respectively. $A \leftarrow B$ means assign B to A.

3. Reducing $(P0)$ and $(Q0)$

The followings are trivial:

Observation 1. $(x^*, y^*) \in OS(P0)$ *satisfies* $x_i^* = 0\ \forall i \in I^{\le}$
Observation 2. $(x^*, y^*) \in OS(Q0)$ *satisfies* $x_j^* = 0\ \forall j \in J^{\ge}$

where the subsets of I and J are given below

$$I^{\le} = \{i \in I \mid a_i/c_i \le b\} \quad I^{>} = \{i \in I \mid a_i/c_i > b\}$$
$$J^{\ge} = \{j \in J \mid d_j/f_j \ge e\} \quad J^{<} = \{j \in J \mid d_j/f_j < e\}$$

By Observation 1 and 2, $(P0)$ and $(Q0)$ reduce to $(P1)$ and $(Q1)$ respectively:

$$\begin{aligned} (P1)\ Max\ & \sum_{i \in I^{>}} a_i x_i + by \\ s.t.\ & \sum_{i \in I^{>}} c_i x_i + y \le K \\ & x_i \in \{0,1\},\ \forall i \in I^{>} \\ & y \ge 0 \end{aligned}$$

$$\begin{aligned} (Q1)\ Min\ & \sum_{j \in J^{<}} d_j x_j + ey \\ s.t.\ & \sum_{j \in J^{<}} f_j x_j + y \ge L \\ & x_j \in \{0,1\},\ \forall j \in J^{<} \\ & y \ge 0 \end{aligned}$$

4. Solving $(P1)$ and $(Q1)$

Lemma 1. *$(x^*, y^*) \in OS(P1)$ is obtained by solving one 0-1 pure integer knapsack problem.*

Proof. For fixed x, $(P1)$ reduces to the following linear programming problem (LP):

$$
\begin{aligned}
(LP)\quad & Max\ by \\
& s.t.\quad y \le K - \sum_{i \in I^{<}} c_i x_i \\
& \qquad\quad y \ge 0
\end{aligned}
$$

Dual of (LP) is

$$
\begin{aligned}
(DP)\quad & Min\ u(K - \sum_{i \in I^{<}} c_i x_i) \\
& s.t.\quad u \ge b
\end{aligned}
$$

$FR(DP)$ has one extreme point, b, and one extreme ray with positive direction. Benders' reformulation[1] of $(P1)$ is given by

$$
\begin{aligned}
(P2)\quad & Max \sum_{i \in I^{<}} a_i x_i + b(K - \sum_{i \in I^{<}} c_i x_i) = bK + (\sum_{i \in I^{<}} (a_i - bc_i)\, x_i \\
& s.t.\quad \sum_{i \in I^{<}} c_i x_i \le K \\
& \qquad\quad x_i \in \{0, 1\},\ \forall i \in I^{<} \qquad ■
\end{aligned}
$$

Lemma 2. *$(x^*, y^*) \in OS(Q1)$ is obtained by solving at most two 0-1 pure integer knapsack problems.*

Proof. For fixed x, $(Q1)$ reduces to the following linear programming problem (LQ):

$$
\begin{aligned}
(LQ)\quad & Min\ ey \\
& s.t.\quad y \ge L - \sum_{j \in J^{<}} f_j x_j \\
& \qquad\quad y \ge 0
\end{aligned}
$$

Dual of (LQ) is

$$
\begin{aligned}
(DQ)\quad & Max\ v(L - \sum_{j \in J^{<}} f_j x_j) \\
& s.t.\quad v \le e \\
& \qquad\quad v \ge 0
\end{aligned}
$$

$FR(DQ)$ has two extreme points, 0 and e. Benders' reformulation of $(Q1)$ is given by

$$
\begin{aligned}
(Q2)\quad & Min\,[\sum_{j \in J^{<}} d_j x_j + max\{0, e(L - \sum_{j \in J^{<}} f_j x_j)\}] \\
& s.t.\quad x_j \in \{0, 1\},\ \forall j \in J^{<}
\end{aligned}
$$

and $OV(Q2) = min\{OV(Q21), OV(Q22)\}$, where

$$(Q21)\quad Min \sum_{j\in J^<} d_j x_j$$
$$s.t. \sum_{j\in J^<} f_j x_j \geq L$$
$$x_j \in \{0,1\},\ \forall j \in J^<$$

$$(Q22)\quad Max \sum_{j\in J^<} (d_j - ef_j) x_j + eL$$
$$s.t. \sum_{j\in J^<} f_j x_j \leq L$$
$$x_j \in \{0,1\},\ \forall j \in J^<$$

$OS(Q1)$ can be obtained by solving at most two 0-1 pure knapsack problems, $(Q21)$ and $(Q22)$. If the optimal solution of either $(Q21)$ or $(Q22)$ satisfies the corresponding knapsack constraint as an equality, there is no need to solve the other knapsack problem. If $OV(Q21) > OV(Q22)$, $y^* = L - \sum_{j\in J^<} f_j x_j^*$. *Otherwise*, $y^* = 0$. ■

5. Examples

We illustrate the solution method with two numerical examples.

5.1 Maximization Problem

$$(P0)\quad Max\ \ 25x_1+5x_2+30x_3+7x_4+12x_5+10y$$
$$s.t.\ \ 10x_1+3x_2+20x_3+8x_4+15x_5+10y \leq 25$$
$$x_i \in \{0,1\},\quad i=1,2,3,4,5$$
$$y \geq 0$$

$(P0)$ reduces to $(P1)$:

$$(P1)\quad Max\ \ 25x_1+5x_2+30x_3+10y$$
$$s.t.\ \ 10x_1+3x_2+20x_3+10y \leq 25$$
$$x_i \in \{0,1\},\quad i=1,2,3$$
$$y \geq 0$$

$(P2)$ is given by

$$(P2)\quad Max\ \ 15x_1+2x_2+10x_3+25$$
$$s.t.\ \ 10x_1+3x_2+20x_3 \leq 25$$
$$x_i \in \{0,1\},\quad i=1,2,3$$

By solving $(P2)$, we obtain the $OS(P0)$: $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)=(1,1,0,0,0,1.2)$.

5.2 Minimization Problem

$$\begin{aligned}(Q0)\quad &Min \quad 10x_1+30x_2+5x_3+18x_4+28x_5+20y\\ &s.t. \quad 10x_1+20x_2+3x_3+6x_4+7x_5+10y\geq 25\\ &\qquad x_i \in \{0,1\},\quad i=1,2,3,4,5\\ &\qquad y\geq 0\end{aligned}$$

$(Q0)$ reduces to $(Q1)$:

$$\begin{aligned}(Q1)\quad &Min \quad 10x_1+30x_2+5x_3+20y\\ &s.t. \quad 10x_1+20x_2+3x_3+10y\geq 25\\ &\qquad x_i \in \{0,1\},\quad i=1,2,3\\ &\qquad y\geq 0\end{aligned}$$

$(Q21)$ and $(Q21)$ are given by

$$\begin{aligned}(Q21)\quad &Min \quad 10x_1+30x_2+5x_3\\ &s.t. \quad 10x_1+20x_2+3x_3\geq 25\\ &\qquad x_i \in \{0,1\},\quad i=1,2,3\\ (Q22)\quad &Min \quad -10x_1-10x_2-x_3+50\\ &s.t. \quad 10x_1+20x_2+3x_3\leq 25\\ &\qquad x_i \in \{0,1\},\quad i=1,2,3\end{aligned}$$

Since $OV(Q21)=40>39=OV(Q22)$ and $(Q22)$ has two alternative optimal solutions, we obtain the $OS(Q0)$: $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, y^*)=(1,0,1,0,0,1.2)$ or $(0,1,1,0,0,0.2)$.

6. Conclusion

Solving any 0-1 mixed integer knapsack or anti-knapsack problem reduces to solving at most two generally smaller pure 0-1 knapsack or anti-knapsack problems with effcient pseudopolynomial time algorithms.

References

[1] Benders, J., "Partitioning Procedures for Solving Mixed Variables Programming Problems," *Numerische Mathematik*, Vol. 4(1962), pp.238-252.

[2] Fayard, D. and G. Plateau, "An Algorithm for the Solution of the 0-1 Knapsack Problem," *Computing*, Vol. 28(1982), pp.269-287.

[3] Nemhauser, F. and L. Wolsey, *Integer and Combinatorial Optimization*, Wiley, 1988.