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Lot-Sizing with Random Yield

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Abstract

Many manufacturing processes involved in the fabrication and assembly of hightech components have highly variable yields that tend to complicate the production control. Under this random yield situation we develop a model to determine optimal input quantity, mean waiting time in the system and variance of waiting time in the system. An example which considers the beta distribution as a yield distribution is given.

1. Introduction

The problem of determining production quantities and their timing, known as lot-sizing, is one common to every production system. Much effort has been focused on solving these problems without considering random yields in production. When yields are random, the production output quantity may differ from the production input quantity. Many manufacturing processes involved in the fabrication of hightech components have highly variable yields that tend to complicate the production control. Specific examples include wafer fabrication and microelectronic assembly. Similar problems also arise in chemical and other process industries. Considerably less research has been done on lot-sizing in the presence of random yield. Shih [15] shows that the optimal production quantity can be found by using a variation of the newsboy model when he assumes that inventory holding costs and shortage costs are linear in their respective arguments, and that the distribution of the fraction defective is invariant with the production level. Gerchak, Palar and Vickson [4] obtained the same result for the profit maximization objective. Ehrhardt and Taube [3] consider a slightly more general problem in which the expected holding and shortage cost function is assumed to be convex.

In the semiconductor industry, it is most important to give the customer what he or she wants in the shortest time possible. One of the biggest problems is dealing with yield varia-

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bility. Manufacturers in this industry would prefer to over-produce than under-produce as customer satisfaction is very important.

The prototypical single-stage, single-period model involves choosing a single lot size to be produced to meet a single demand by minimizing the sum of expected inventory holding cost and shortage cost. In this paper we shall investigate the implications of yield randomness (output variability) for lot-sizing in production system. In section 2 we shall define and analyze a model to investigate the effect of random yields on lot-sizing. In section 3 we shall consider an example. We shall conclude and discuss directions for future research in section 4.

2. Model Development

In this section the analytical formulation of the problem is presented. We first introduce a set of notations which is used throughout this paper. The sign ~ is used to signify 'with distribution function.'

D=Demand per period which is assumed as constant.

P=Yield rate random variable~F(p).

p_n=Yield rate in period n (fraction of good product).

I_n=End-of-period-inventory in period n.

a=Yield adjustment factor.

α=Service level (probability of satisfying demand).

h=inventory holding cost per unit per period

π=shortage cost per unit per period.

Q_n=Input quantity of period n.

Here we assume that the production lead time is less than one period with probability 1, or that the feedback about the actual yield is known with certainty within one period. So we consider production policy of the form

$$Q_n = a(D - I_{n-1})$$

which is equivalent to requiring that

$$\text{Prob}\,[I_{n-1} + p_n Q_n \geq D] = \alpha$$

where $\alpha = 1 - F(a)$. We also assume that $D - I_{n-1}$ is always non-negative.

If we start with $I_0 = 0$, then it is easy to show that

$$I_n = (1 - ap_n)(I_{n-1} - D)$$

and

$$\lim_{n\to\infty} E(I_n) = \frac{D(aE(P)-1)}{aE(P)}$$

if $|1-aE(P)| < 1$, which is equivalent to $0<a<\frac{2}{E(P)}$ since $E(P)>0$. This condition simply means that the yield adjustment factor is less than twice the factor required to compensate for the average yield loss. In most cases, this is a very mild condition, because capacity constraints are often much more restrictive. Throughout the remainder of this paper, we shall simply assume that this condition holds.

Taking the limit as $n\to\infty$, we find that

$$I \to \frac{-D(1-aP)}{aP} \text{ in distribution.} \tag{2.1}$$

Thus,

$$E(I) = \left(\frac{-D}{a}\right)\left(E(P^{-1})-a\right)$$

and

$$E(I^2) = \left(\frac{D^2}{a^2}\right)\left(E(P^{-2}) - 2aE(P^{-1}) + a^2\right)$$

Therefore,

$$\begin{aligned} Var(I) &= \left(\frac{D^2}{a^2}\right)\left(E(P^{-2}) - E(P^{-1})^2\right) \\ &= \left(\frac{D^2}{a^2}\right) Var(P^{-1}) \end{aligned}$$

Note that $E(I)$ is increasing with a and $Var(I)$ is decreasing with a. Therefore, because we are interested in selecting the value of a to minimize the expected holding and shortage costs, an appropriate tradeoff could be accomplished by getting the distribution of I from distribution of P as follows:

Let the pdf of I and P be $g(i)$ and $f(p)$ respectively. From(2.1), we get the pdf of I from the following relation (see Hogg and Craig [5])

$$g(i) = f(w(i)) \left|\frac{dp}{di}\right|$$

where $p=w(i)$. By substituting the appropriate quantity to the abvoe relation, we get

$$g(i) = f\left(\frac{-D}{a(i-D)}\right)\frac{D}{a(i-D)^2}$$

Thus, the total expected cost (TEC) per period is

$$TEC = h\int_0^{\infty} ig(i)di - \pi\int_{-\infty}^{0} ig(i)di$$

which is equivalent to

$$TEC = (h + \pi) \int_0^x ig(i)\, di - \pi E(I)$$

By substituting the expression of E(I) and g(i) to TEC we get

$$TEC = \frac{D}{a} \left\{ -\pi(a - E(P^{-1})) + (h+\pi) \int_0^x \frac{i}{(i-D)^2} f\left(\frac{D}{a(D-i)}\right) di \right\}$$

From the expression of TEC, we find a which minimize TEC by setting the derivative of TEC with respect to a as zero, i.e.,

$$\frac{dTEC}{da} = -\frac{D\pi}{a^2} E(P^{-1}) - \frac{D}{a^2}(h+\pi) \int_0^x \frac{i}{(i-D)^2} \left\{ f\left(\frac{D}{(D-i)}\right) + f'\left(\frac{D}{a(D-i)}\right) \frac{D}{a(D-i)} \right\} di = 0$$

Which is equivalent to find a satisfying the following equation

$$\frac{\pi}{h+\pi} E(P^{-1}) = -\int_0^x \frac{i}{(i-D)^2} \left\{ f\left(\frac{D}{a(D-i)}\right) + f'\left(\frac{D}{a(D-i)}\right) \frac{D}{a(d-i)} \right\} di$$

We now show how to find the mean waiting time in the queue and the variance of waiting time in the queue from the information about the first and second moments of input quantity. The production system that we consider in this paper corresponds to D/G/1 queuing system since arrival stream (demand) is constant and service time depends on the batch size, i.e., input quantity. We can use the results available for G/G/1 case to get the results for D/G/1 case.

Marshall [9] suggests the lower bound for the mean waiting time in the queue for G/G/1 as

$$E(W_q)_M = \frac{\rho^2(1+C_s^2)}{1+\rho^2 C_s^2} \left\{ \frac{C_a^2 + \rho^2 C_s^2}{2\lambda(1-\rho)} \right\}$$

where $\rho = \lambda E[s]$

$C_a^2 = \frac{Var[IAT]}{E[IAT]^2}$ = the coefficient of variation of interarrival time (IAT)

C_s^2=the coefficient of variation of service time defined similarly as C_a^2.

Kingman [16] also suggests the upper bound for the mean waiting time in the queue for G/G/1 as

$$E(W_q)_K = \frac{C_a^2 + \rho^2 C_s^2}{2\lambda(1-\rho)}$$

Thus, we get the bounds for the mean waiting time in the queus as

$$E(W_q)_M \le E(W_q) \le E(W_q)_K$$

So the bounds for the mean waiting time in the system are given by adding the serviced time to the mean waiting time in the queue.

Kleinrock [6] shows the bounds for the variance of waiting time in the queue as

$$Var[s] \le Var[W_q] \le Var[IAT] + 2Var[s] - \frac{2}{\lambda}E(W_q)_M(1-\rho)$$

The bounds for the variance of waiting time in the system are the sum of the variance of service time and the variance of waiting time in the queue since service time and waiting time in the queue are independent.

E[s] and Var[s] is given from the first and second moments of input quantity as follows since service time is proportional to the input quantity.

$$E[s] = E[Q]$$
$$Var[s] = E[s^2] - E[s]^2 = E[Q^2] - E[Q]^2$$

Here, the moments of input quantity are derived from the moments of inventory and input quantity (Q) follows a(D−I) in distribution

$$E[Q] = aD - aE[I] = DE[P^{-1}]$$

and similarly,

$$E[Q^2] = a^2E[(D-I)^2] = D^2E[P^{-2}]$$

Hence, Var[s] is given as

$$Var[s] = D^2(E[P^{-2}] - E[P^{-1}]^2) = D^2\, Var[P^{-1}]$$

3. Example

We consider the beta distribution as yield rate distribution. f(p) has the following form.

$$f(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, p^{\alpha-1}(1-p)^{\beta-1}$$

where mean is $\frac{\alpha}{\alpha+\beta}$ and variance is $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

We get the distribution of inventory from the formula in section 2 as

$$g(i) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{-D}{a(i-D)}\right)^{\alpha-1}\left(1+\frac{D}{a(i-D)}\right)^{\beta-1}\frac{D}{a(i-D)^2},\quad -\infty < i < D-\frac{D}{a}$$

The total expected cost is, thus, given by

$$TEC = \frac{D}{a}\left\{-\pi(a-E(P^{-1})) + (h+\pi)\int_0^{D-\frac{D}{a}} \frac{i}{(i-D)^2}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\left(\frac{D}{a(D-i)}\right)^{\alpha-1}\left(1-\frac{D}{a(D-i)}\right)^{\beta-1}di\right\}$$

From this equation we can find the value of a which minimizes TEC by setting $\frac{dTEC}{da}=0$ which corresponds to find a from the following equation

$$\frac{\pi}{h+\pi}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{\alpha+\beta-1}{\alpha-1}=$$

$$-\int_0^x \frac{i}{(i-D)^2}\left(\frac{D}{a(D-i)}\right)^{\alpha-2}\left(1-\frac{D}{a(D-i)}\right)^{\beta-2}\left\{\frac{D\alpha}{a(D-i)}-\left(\frac{D}{a(D-i)}\right)^2(\alpha+\beta-1)\right\}di$$

We can find a by numerical integration

a. case 1

$\alpha=6,\ \beta=2,\ h=1,\ \pi=4,\ D=0.65$

This case corresponds to beta distribution with mean of 0.75 and utilization factor is 91%.
In this case we get the value of a as 1.7.

b. case 2

$\alpha=6,\ \beta=2,\ h=1,\ \pi=4,\ D=0.4$

This case corresponds to beta distribution with mean of 0.75 and utilization factor is 56%.
In this case we have 1.5 as value of a.

The mean waiting time in the queue and the variance of waiting time in the queue for D/G/1 considered in this paper are calculated as follows:
Let's first calculate the $E[P^{-n}]$

$$\begin{aligned}E[P^{-n}] &= \int_0^1 p^{-n}f(p)dp\\ &= \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-n-1}(1-p)^{\beta-1}dp\\ &= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha-n)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha-n+\beta)}\\ &= \frac{(\alpha+\beta-1)!(\alpha-n-1)!}{(\alpha-1)!(\alpha-n+\beta-1)!}\end{aligned}$$

Therefore, $E[P^{-1}]=\frac{\alpha+\beta-1}{\alpha-1}$ and $E[P^{-2}]=\frac{(\alpha+\beta-1)(\alpha+\beta-2)}{(\alpha-1)(\alpha-2)}$

In the D/G/1 case, $C_a^2=0$ and

$$\begin{aligned}C_s^2 &= \frac{Var[s]}{E[s]^2}=\frac{D^2Var[P^{-1}]}{(DE[P^{-1}])^2}\\ &=\frac{Var[P^{-1}]}{E[P^{-1}]^2}=\frac{\beta}{(\alpha+\beta-1)(\alpha-2)}\end{aligned}$$

Thus, the mean waiting time in the queue has as its bounds

$$\frac{\rho^2\left(1+\frac{\beta}{(\alpha+\beta-1)(\alpha-2)}\right)}{1+\rho^2\frac{\beta}{(\alpha+\beta-1)(\alpha-2)}}\left\{\frac{\rho^2\frac{\beta}{(\alpha+\beta-1)(\alpha-2)}}{2\lambda(1-\rho)}\right\}\leq E(W_q)\leq\frac{\rho^2\frac{\beta}{(\alpha+\beta-1)(\alpha-2)}}{2\lambda(1-\rho)}$$

and the variance of waiting time in the queue has as its bounds

$$D^2\frac{(\alpha+\beta-1)\beta}{(\alpha-1)^2(\alpha-2)} \le Var(W_q)$$

$$\le 2D^2\frac{(\alpha+\beta-1)\beta}{(\alpha-1)^2(\alpha-2)\lambda} - \frac{2}{\lambda}\frac{\rho^2(1+\frac{\beta}{(\alpha+\beta-1)(\alpha-2)})}{1+\rho^2\frac{\beta}{(\alpha+\beta-1)(\alpha-2)}}\left\{\frac{\rho^2\frac{\beta}{(\alpha+\beta-1)(\alpha-2)}}{2\lambda(1-\rho)}\right\}(1-\rho)$$

a. case 1

$\alpha=6,\ \beta=2,\ D=0.65$

In this case, parameter α and β correspond to beta distribution with mean of 0.75. Utilization factor is 91%(High utilization). We get the mean waiting time in the queue and variance of waiting time in the queue as follows:

$0.275 \le E(W_q) \le 0.329$

$0.059 \le Var(W_q) \le 0.069$

We also get the mean waiting time in the system and variance of waiting time in the system from the above values as follows

$1.185 \le E(W_s) \le 1.239$

$0.118 \le Var(W_s) \le 0.128$

b. case 2

$\alpha = 6,\ \beta = 2,\ D = 0.4$

In this case, parameter α and β correspond to same beta distribution with mean of 0.75. However, utilization factor is 56%(Low utilization). We obtain the mean waiting time in the queue and variance of waiting time in the queue as follows:

$0.008 \le E(W_q) \le 0.025$

$0.022 \le Var(W_q) \le 0.037$

Similarly, we derive at the values for the mean waiting time in the system and variance of waiting in the system as

$0.568 \le E(W_s) \le 0.585$

$0.044 \le Var(W_q) \le 0.059$

4. Conclusion

We find that the value of variable "a" decreases as the demand rate per period decreases. We also show that the mean waiting time in the queue and variance of waiting time in the queue increase as the demand rate per period increases. Note that the value of variable "a" does not affect the mean waiting time in the queue nor the variance of waiting time in the queue.

Further research needs to incorporate other realistic factors, such as setup costs, demand uncertainty, possible rework, multiple production batches and multi-period into the current model.

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