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A Study on the Phenomenon of Natural Zoning under COL Storage Policy

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Abstract

In this paper, we first examine the stochastic behavior associated with storage/retrieval process under COL policy, and give analytical results such as the limiting distributions of the number of items in the system and the expected travel distance. We also investigate the phenomenon of natural zoning, that is, the tendency of similar items to group themselves together, when two types of items with different turnover rates are stored and retrieved. Natural zoning refers to this grouping occurring under the seemingly unbiased policy COL. We show that zoning can occur naturally with batch arrivals to the storage system.

1. Introduction

Suppose that we have two types of items (high and low turnover) to store in a warehouse. We ask whether the high and low turnover items tend to form zones naturally when items are always stored in the closest oplen location. A survey [6] shows that roughly half of the managers of storage systems believe that natural zoning occurs while the rest believe that the items are distributed at random. This natural zoning would be beneficial if the high turnover items tended to zone themselves near the front or would be detrimental if they migrate towards the rear. (If the case is the latter, it will be needed to find some storage policies that induce the former.) Throughout this paper, some assumptions such as — sufficiently large number of identical storage locations, a unique I/O point at the left corner of the storage rack, only one item storage in a location—are made. We also assume a linear storage with $d_n = n$ where the n^{th} location is d_n units from the I/O point.

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2. COL Storage Policy

Assume that items arrive according to a Poisson process with rate λ and each arriving item is stored in the first available (empty) location, and each stored item stays in a location exponentially distributed length of time with parameter μ. Let $N(m,t)$ be the number of occupied locations amongst m locations through 1 to m. Then, the stationary distribution, which is known as Erlang's first formula, can be readily obtained (see Gross and Harris [2])as:

$$P_m(i)=\frac{\rho_i / i!}{\sum_{k=0}^{m} \rho^k / k!}$$

where $p=\lambda/\mu$.

Let $D(COL)$ denote the expected travel distance in the COL storage policy. Let L_n be the position of the n^{th} open location. Then, by PASTA,

$$D(COL)=E[L_1]=\sum_{m=0}^{\infty} Pr\{L_1>m\}=\sum_{m=0}^{\infty} P_m(m)=\sum_{m=0}^{\infty}\frac{\rho^m/m!}{\sum_{k=0}^{m}\rho^k/k!}=\sum_{m=0}^{\infty} B(m,\rho). \qquad (1)$$

Now let us consider the case of two types of items. Assume that two types of items, type h and type l, arrive according to Poisson processes with rate λ_h and λ_l respectively and each arriving type $h[l]$ item is stored in the first available location and stays in the location exponentially distributed length of time with parameter $\mu_h[\mu_l]$ respectively.

Let $N^h(m, t)$ and $N^l(m, t)$ denote number of type h and tyupe l items among first m locations respectively.

Then,

$$N(m, t)=N^h(m, t)+N^l(m, t)$$

An elementary calculation shows that the stationary distribution of $(N^h(m, t), N^l(m, t))$ has the product form:

$$P_m(i, j)=\frac{\rho_h^i}{i!}\cdot\frac{\rho_l^j}{j!}\cdot P_m(0, 0) \qquad (2)$$

where

$$P_m(0,0)=\left[\sum_{i+j\leq m}\frac{\rho_h^i}{i!}\cdot\frac{\rho_l^j}{j!}\right]^{-1}$$

with $\rho_h=\lambda_h/\mu_h$, $\rho l=\lambda_l/\mu_l$.

Then the expected travel distance

$$D(COL)=E[L_1]=\sum_{m=0}^{\infty} Pr\{L_1>\mathrm{m}\}=\sum_{m=0}^{\infty} Pr\{N^h(m)+N^l(m)=m\}$$

$$= \sum_{m=0}^{\infty} \sum_{i=0}^{m} Pr\{N^h(m)=i,\ N^l(m)=m-i\} = \sum_{m=0}^{\infty} \left[\frac{\sum_{i=0}^{m} \rho_h^i \rho_l^{m-i} / i!(m-i)!}{\sum_{i=0}^{m} \sum_{j=0}^{m-1} \rho_h^i \rho_l^j / i!j!} \right] ,$$

$$= \sum_{m=0}^{\infty} \left[\frac{\sum_{i=0}^{m} \rho_h^i \rho_l^{m-i} / i!(m-i)!}{\sum_{k=0}^{m} \sum_{i=0}^{k} \rho_h^i \rho_l^{k-i} / i!(k-i)!} \right] .$$

Since

$$\sum_{i=0}^{k} \rho_h^i \rho_l^{k-i} / i!(k-i)! = (\rho_h + \rho_l)^k / k!,$$

we have

$$D(COL) = \sum_{m=0}^{\infty} \left[\frac{(\rho_h + \rho_l)^m / m!}{\sum_{k=0}^{m} (\rho_h + \rho_l)^k / k!} \right] = \sum_{m=0}^{\infty} B(m,\ \rho_h + \rho_l) \tag{3}$$

Remark 1. *Note that the expected travel distance D(DOL) of two types of items is reduced to that of one type of items with different offered load* $\rho_h + \rho_l$ *instead of* ρ.

Remark 2. *This result also follows from the insensitivity of the M/G/m/m. That is, the Erlang's loss formula for M/M/m/m queue is still valid for M/G/m/m queue, and we can consider the two types as one type with a hyperexponential service time.*

The above result can be extended in the same way to obtain

$$D(COL) = \sum_{m=0}^{\infty} B(m, \rho_1 + \rho_2 + \cdots + \rho_n) \tag{4}$$

for n types of items.

3. Natural Zoning

3.1 Definitions of Natural Zoning

There are several possible definitions of natural zoning. We could say that natural zoining occurs if one or more of the following conditions are satisfied.

(1) The joint stationary distribution of the numbers of items for each type does not have a product form, that is,

$$P_m(i, j) \neq K \cdot P^l_m(i) P^l_m(j),\ i+j \leq m$$

for any m, where K is a constant.

(2) For any m,

$$Pr\{N^h(m)=i \mid N^h(m)+N^l(m)=\mathrm{n}\} \neq \frac{n!}{i!(n-i)!}p^i(1-p)^{n-i}.$$

That is, zoning does not occur if given the total number of items stored in the first m locations, the number of type h has a binomial distribution..

(3) For any m,

$$E\left[\frac{N^h(m)}{N^h(m)+N^l(m)} \mid N^h(m)+N^l(m)>0\right]$$

does not have a constant value.

Each of the above three definitions are concerned with the items grouping themselves together. However, in some sense this grouping is only a symptom, and the important question is whether different items have different expected travel distances. Hence, although we discuss all definitions, we will focus most of our attention on the following definition.

(4) Zoning occurs if the expected travel distance of type h items is not the same as that of type l items, that is,

$$D^h(COL) \neq D^l(COL)$$

Let us again consider the COL storage policy with two types of items with Poisson arrivals as mentioned in Section 2. Repeating Equatuion (2),

$$P_m(i, j)=\frac{\rho_h^i}{i!} \cdot \frac{\rho_l^j}{j!} \cdot P_m(0,0)$$

where

$$P_m(0, 0)=\left[\sum_{i+j\leq m} \frac{\rho_h^i}{i!} \cdot \frac{\rho_l^j}{j!}\right]^{-1}, \ i+j\leq m.$$

Then

$$P_m(i, j)=K \cdot P^h_m(i)P^l_m(j)$$

where

$$K=\frac{\left(\sum_{i=0}^{m}\frac{\rho_h^i}{i!}\right)\left(\sum_{j=0}^{m}\frac{\rho_l^j}{j!}\right)}{\sum_{i+j\leq m}\frac{\rho_h^i}{i!} \cdot \frac{\rho_l^j}{j!}},$$

$$P^h_m(i)=\frac{\rho_h^i}{i!}\left[\sum_{i=0}^{m}\frac{\rho_h^i}{i!}\right]^{-1},$$

$$P^l_m(j)=\frac{\rho_l^j}{j!}\left[\sum_{j=0}^{m}\frac{\rho_l^j}{j!}\right]^{-1}.$$

Hence definition (1) would be satisfied.

On the other hand,

$$\Pr\{N^h(m)=i \mid N^h(m)+N^l(m)=n\}, \quad 0\le i \le n\le m$$

$$=\frac{\Pr\{N^h(m)=i,\ N^l(m)=n-i\}}{\sum_{i=0}^{n} Pr\{N^h(m)=i,\ N^l(m)=n-i\}}={}_nC_i\left(\frac{\rho_h}{\rho_h+\rho_l}\right)^i\left(\frac{\rho_l}{\rho_h+\rho_l}\right)^{n-i}$$

Then, the conditional distribution turns out to be binomial, and hence Definition (2) is satisfied. Furthermore,

$$E\left[\frac{N^h(m)}{N^h(m)+N^l(m)}\right]=E\left[E\left[\frac{N^h(m)}{N^h(m)+N^l(m)} \mid N^h(m)+N^l(m)\right]\right]=p$$

where $p=\frac{\rho_h}{\rho_h+\rho_l}$ Hence Definition (3) is satisfied.

From *PASTA*, Definition (4) is satisfied. Hence, for Poisson arrival and two types of items, zoning does not occur under any of the definitions.

3.2 Batch Arrivals

Assume that type $h[l]$ batches arrive according to a compound Poisson process with rate λ_h $[\lambda_l]$ and random variable $X[Y]$ denotes batch size of type $h[l]$ item, and each arriving type $h[l]$ item is stored for an exponentially distributed length of time with parameter $\mu_h[\mu_l]$ respectively.

Now we turn our attention to Definition (4) to determine whether the travel time of the type h and l items are different. First we need to define *stochastically larger* [8].

Definition 1. *A random variable X is stochastically larger than another random variable Y, expressed as $X \ge_{st} Y$, if* $\Pr\{X > c\}\ge \Pr\{Y > c\}$ *for all c.*

It is well known that if $X \ge_{st} Y$ and $E[X]$ and $E[Y]$ exist, then $E[X] \ge E[Y]$. Also, if $X \ge_{st} Y$, then there exists a sample space with random variables X^* and Y^* which have the same marginal distributions as X and Y and such that

$$\Pr\{X^*\ge Y^*\}=1.$$

Now consider the storage/retrieval process with two types of items arriving according to compound Poisson processes and linear storage. Let $X >_{st} Y$ mean that $X \ge_{st} Y$ but $X\ne Y$.

Theorem 1. *If $X>_{st} Y$, then*

$$D^h(COL)> D^l(COL),$$

that is, natural zoning occurs. If $X <_{st} Y$, *then*

$$D^h(COL) < D^l(COL).$$

Proof. Let L_n the n^{th} open location. Let the random variable $D^h(COL)[D^l(COL)]$ denote the average storage distance when a batch of type $h[l]$ items are stored respectively. Then, by PASTA and the fact that $d_n = n$, we have

$$D^h(COL) = E\left[\frac{\sum_{n=1}^{X} L_n}{X}\right]$$

$$D^l(COL) = E\left[\frac{\sum_{n=1}^{Y} L_n}{Y}\right]$$

First suppose that $X >_{st} Y$. Using the coupling argument, we can assume that $\Pr\{X \geq Y\} = 1$ and $\Pr\{X = Y\} < 1$. Obviously $L_n < L_{n+1}$. Hence,

$$\begin{aligned}
D^h(COL) - D^l(COL) &= E\left[\frac{\sum_{n=1}^{X} L_n}{X} - \frac{\sum_{n=1}^{Y} L_n}{Y} \mid X > Y\right] \Pr\{X > Y\} \\
&= E\left[\frac{\sum_{n=1}^{Y} L_n + \sum_{n=Y+1}^{X} L_n}{X} - \frac{\sum_{n=1}^{Y} L_n}{Y} \mid X > Y\right] \Pr\{X > Y\} \\
&\geq E\left[\frac{(\sum_{n=1}^{Y} L_n + (X-Y)L_Y}{X} - \frac{\sum_{n=1}^{Y} L_n}{Y} \mid X > Y\right] \Pr\{X > Y\} \\
&= E\left[\frac{(X-Y)(YL_Y - \sum_{n=1}^{Y} L_n)}{XY} \mid X > Y\right] \Pr\{X > Y\} > 0.
\end{aligned}$$

The case that $X <_{st} Y$ follows similarly. ■

The batch size of high turnover items is typically larger than that of low turnover items. The above theorem implies that unfavorable zoning naturally occurs if the COL storage policy is employed and high turnover items have stochastically larger batches.

Although we do not obtain the stationary distribution of the numbers of items in the system described in this section, we could show that the stationary distribution has a product form [5]. However, Theorem 1 shows that if the batch size of one type is stochastically larger than that of the other type, a natural zoning occurs. Hence, in general, we can say that a product form stationary distribution does not guarantee that natural zoning does not occur.

Corollary 1. If $X \overset{d}{=} Y$, *then*

$$D^h(COL) = D^l(COL)$$

that is, natural zoning does not occur.

Proof. Regardless of type of items, when a batch arrives, the n^{th} item of the batch will be stored in the n^{th} open location. Since batch size distributions are identical, the expected travel distance of the n^{th} item in a batch (whether type h or l) will be the same as $E[L_n]$, n=1,2,⋯. ■

As a special case of the above Corollary 1, if $X = Y = 1$, that is, if arrival processes are Poisson processes, then natural zoning does not occur.

We notice that the *occurrence of natural zoning* depends on the arrival process but does not depend on the demand process (retrieval process).

4. Conclusion

We showed that in batch arrivals, if type h items have stochastically larger batch size than that of type l items, then the expected travel distance of type h is larger than that of type l. However, usually the batch size of high turnover items is larger than that of low turnover items, and hence, unfavorable zoning can occur naturally if the COL Storage is applied. This fact leads us to consider some storage policies which will induce favorble zonings.

Most of the results in this paper might be extended to the case of square storage rack with Tchebychev travel. Notice that even if square storage rack is assumed, the number of items in the system is not changed and square rack can be expanded to linear storage.

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