Journal of the Korean
OR/MS Society
Vol. 17, No. 1, April 1992

A Simple Extension of the Global Optimality Condition for Lagrangean Relaxation

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Abstract

A slight extension of the classical saddle point and the global optimality condition has been discussed relative to some algorithmic implications. It also involves an economic interpretation which shows satisfying, rather than optimizing, decision making behavior under bounded rationality.

1. Introduction

Since the pioneering paper of Everett [2] the Lagrangean relaxation has been widely used to optimize various mathematical programming problems with complicationg side constraints. The well known *global optimality condition* (Shapiro [7, p.144], Geoffrion [4]) has provided a theoretical foundation. However it has been seldom observed in practice, mainly, due to the existence of positive duality gap in many real problems. This is particularly true of almost all discrete optimization problems to which the Lagrangean technique has been applied (Geoffrion [5], Fisher [3]) with salient success.

The main emphasis of studies on the problem of duality gap, till now, has been put on removing the duality gap by using various nonlinear pricing methods instead of multipliers, which has been well summarized by Tind and Wolsey [10]. However it should be noted that nonlinear pricing itself is hard to handle for practical purposes and that the ordinary Lagrangean relaxation is still being popularly used even without proper theoretical foundation. It has been taken for granted implicitly that no useful result about optimality can be derived under positive duality gap.

This paper takes a different view. The duality gap is allowed, and a simple extension of the global optimality condition has been presented using ordinary Lagrangean function. It is in fact

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a direct extension of the classical *saddle point* and optimality condition. It suggests a new insight on the use of Lagrangean relaxation, and justifies theoretically the frequently used heuristic procedures of manipulating a good feasible solution of the original problem from an optimal solution of the Lagrangean subproblem.

The associated economic interpretation shows satisfying (Simon [8]), rather than optimizing, decision making behaviors.

2. ε-saddle point

Let X and Y be any nonempty sets and $L(x,y)$ be a real valued function on $X\times Y$.

Definition 1. *Let ε, ε_1, ε_2 be nonnegative numbers with $\varepsilon_1+\varepsilon_2=\varepsilon$. A pair $(\bar{x}, \bar{y}) \in X \times Y$ is an ε-saddle point for $L(x, y)$ if*

$$L(x, \bar{y})-\varepsilon_1\leq L(\bar{x}, \bar{y})\leq(\bar{x}, y)+\varepsilon_2 \qquad \forall x\in X, \quad y\in Y.$$

The conventional saddle point corresponds to a zero-saddle point in the definition above. The following economic interpretation is possible for small values of ε : Consider a *two person zero-sum game* where X, Y denote the set of strategies of *Player 1* and *Player 2* respectively. Let $L(x, y)$ be the *payoff function* of Player 1. Then an ε-saddle point is a pair of satisfactory, rather than optimal, strategies of the two players. It can be a stable solution of the game if the players are not completely aware of the strategies and the payoff structure.

Theorem 1. *Define $M(y)=\max_{x\in X} L(x, y)$* and *$m(x)=\min_{y\in Y} L(x, y)$.*
Then $(\bar{x}, \bar{y})\in X\times Y$ is an ε-saddle point if and only if $M(\bar{y})-m(\bar{x})\leq\varepsilon$.

Proof. Since $M(\bar{y})-m(\bar{x})=(M(\bar{y})-L(\bar{x}, \bar{y})+(L(\bar{x}, \bar{y})-m(\bar{x}))$, the result is clear. ■

Unlike the conventional saddle point, the ε-saddle point exists for any function $L(x, y)$ for a proper value of ε without any regularity conditions. It could be found by (approximately) optimizing $M(y)$ and $m(x)$ respectively, probably with known algorithms.

3. Global ε-optimality condition

Consider the following mathematical programming problem

$$\text{(P)} \quad \max\{f(x) \mid g_i(x)\leq 0,\ i=1, \cdots, m,\ x\in X\}$$

where X is a nonempty subset of R^n(the Euclidean n-dimensional space), and $f(x)$, $g_i(x)$ are real valued functions on X.

Suppose(P) can be solved easily with known algorithms without the m constraints $g_i(x) \leq 0$. Define

$$L(x, y) = f(x) - \sum_{i=1}^{m} y_i g_i(x) \text{ on } X \times Y, \text{ where } Y = \{y \in R^m \mid y \geq 0\}.$$

A natural Lagrangean relaxation [5] with a multiplier vector $y \in Y$ is

$$(\mathrm{R}_y) \qquad M(y) = \max_{x \in X} L(x, y).$$

The Lagrangean dual problem is

$$(\mathrm{D}) \qquad \min_{y \in Y} M(y).$$

Definition 2. *Let ε, ε_1, ε_2 be nonnegative numbers with $\varepsilon_1 + \varepsilon_2 = \varepsilon$. A pair $(\bar{x}, \bar{y}) \in X \times Y$ satisfies the global ε-optimality condition if*

(i) *$\bar{x}$ is an ε_1-optimal solution of $(R_{\bar{y}})$, i.e., $L(\bar{x}, \bar{y}) \geq M(\bar{y}) - \varepsilon_1$,*

(ii) *$g_i(\bar{x}) \leq 0$, $i = 1, \cdots, m$, and*

(iii) *$\sum_{i=1}^{m} \bar{y}_i(-g_i(\bar{x})) \leq \varepsilon_2$.*

The above definition is a direct extension of the global optimality condition which corresponds to the specific instance of $\varepsilon = 0$.

Theorem 2. *Suppose $\bar{x}$ is feasible to (P)* and *$\bar{y} \in Y$*. Then *$(\bar{x}, \bar{y})$ satisfies the global ε-optimality condition if and only if $M(\bar{y}) - f(\bar{x}) \leq \varepsilon$.*

Poof. Since $M(\bar{y}) - f(\bar{x}) = M(\bar{y}) - L(\bar{x}, \bar{y}) + L(\bar{x}, \bar{y}) - f(\bar{x}) = M(\bar{y}) - L(\bar{x}, \bar{y}) - \sum_{i=1}^{m} \bar{y}_i g_i(\bar{x})$, the result is clear. ■

Corollary 2.1. *A pair $(\bar{x}, \bar{y}) \in X \times Y$ is an ε-saddle point for $L(x, y)$ if and only if it satisfies the global ε-optimality condition.*

Proof. Since $m(\bar{x}) = \min_{y \in Y} L(\bar{x}, y) = \begin{cases} f(\bar{x}), & \text{if } \bar{x} \text{ is feasible to (P)} \\ -\infty, & \text{otherwise,} \end{cases}$

Theorems 1 and 2 directly leads to the above result. ■

Many practical algorithms using Lagrangean relaxation terminate when the gap between the upper bound and the lower bound of the optimal value has been within some predetermined error bound ε. In this light, Theorem 2 has revealed that the global ε-optimality condition actually should be satisfied for many usual algorithms to terminate.

Of course the usefulness of the global ε-optimality condition will increase as the Lagrangean duality gap decreases. Although we can in general narrow down the duality gap by reducing the number of relaxed constraints, this will restrict the availability of easy exact algorithms for solving the Lagrangean subproblems. However we may benefit from using some easy heuristic algorithms, instead of exact ones, since an approximate optimal solution of (P) will be also found as an approximate optimal solution of ($R_{\bar{y}}$) if only $\bar{y}$ is (near) optimal to (D). It also should be noted that an ε-optimal solution $\bar{x}$ of ($R_{\bar{y}}$) produces an ε-subgradient (Dem'yanov and Vasil'ev [1, p.77], $-g(\bar{x})$, of the convex function $M(y)$ at $\bar{y}$, which can be useful to minimize $M(y)$.

The global ε-optimality condition also gives a theoretical justification to many successful heuristic attempts, frequently used hitherto, to obtain a good feasible solution of (P) by slightly modifying an optimal solution of a Lagrangean subproblem. In particular, once we have obtained an (opproximate) optimal multiplier vector $\bar{y}$ of (D), Theorem 2 recommends us to examine as many approximate optimal solutions of ($R_{\bar{y}}$), beyond a single optimal one, as we can with ease. This additional effort might reduce the necessity of entering into tedious branch and bounds, by detecting out a satisfactory solution of (P).

4. Economic interpretation in linear programming with mixed variables

Consider the following problem to maximize an economic criterion cx,

$$\text{(P)} \qquad \max\{cx \mid Ax \leq b,\ x \in X\}$$

where x is a vector of levels of n activities, $Ax \leq b$ is the set of constraints confining the availability of m resources for the activities. Let $X = \{x \in R^n \mid x \geq 0,\ x_1, \cdots, x_k$ are integers. $(k \leq n)\}$. Note that (P) becomes a pure integer program if $k=n$, and simply a linear program if $k=0$. Define $L(x, y) = cx + y(b - Ax)$ on $X \times Y$, where $Y = \{y \in R^m \mid y \geq 0\}$. The following result is direct from Definition 2.

Theorem 3. *A pair $(\bar{x}, \bar{y}) \in X \times Y$ satisifes the global ε-optimality condition if and only if*

(i) $c-\bar{y}A \leq 0$

(ii) $(\bar{y}A-c)\bar{x} \leq \varepsilon_1$

(iii) $A\bar{x} \leq b$

(iv) $\bar{y}(b-A\bar{x}) \leq \varepsilon_2$

for some nonnegative numbers ε_1, ε_2 *with* $\varepsilon_1+\varepsilon_2=\varepsilon$.

Suppose $\bar{x}$ is the vector of chosen levels of activities and $\bar{y}$ is the vector of imputed prices of resources. For small values of ε, the following interpretations come from Theorem 3.

(i) The prices give no net profit for any activity.

(ii) If net loss arises from an activity, we either give up the activity, or do it no greater than a reasonable level.

(iii) The chosen set of activities is feasible.

(iv) We make use of either full, or a great majority of valuable resources. The unused resources have little value, if any.

The above interpretation represents a satisfying [8], not necessarily optimizing, condition for decision making. This better reflects real decision making behaviors under bounded rationality (Simon [9, p.61]), than the well known similar interpretation in linear programming (Koopmans [6, p.216]) which described the optimal behavior.

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