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THE ESSENTIAL POINT SPECTRUM OF A REGULAR OPERATOR

WOO YOUNG LEE, HONG YOUL LEE AND YOUNG MIN HAN

In [5] it was shown that if $T \in \mathcal{L}(X)$ is regular on a Banach space X, with finite dimensional intersection $T^{-1}(0) \cap T(X)$ and if $S \in \mathcal{L}(X)$ is invertible, commute with T and has sufficiently small norm then $T - S$ is upper semi-Fredholm, and hence *essentially one-one*, in the sense that the null space of $T - S$ is finite dimensional ([4] Theorem 2; [5] Theorem 2). In this note we extend this result to incomplete normed spaces.

Throughout this note suppose X and Y are normed spaces, write $\mathcal{L}(X, Y)$ for the set of all bounded linear operators from X to Y, and abbreviate $\mathcal{L}(X, X)$ to $\mathcal{L}(X)$. Recall ([2],[3]) that $T \in \mathcal{L}(X, Y)$ is said to be *bounded below* if there is $k > 0$ for which

$$\|x\| \le k\|Tx\| \quad \text{for each } x \in X,$$

is said to be *open* if there is $k > 0$ for which

$$y \in \{Tx : \|x\| \le k\|y\|\} \quad \text{for each } y \in Y$$

and is said to be *relatively open* if its truncation $T^\wedge : X \longrightarrow T(X)$ is open. Thus bounded below is just relatively open one-one. The mapping $\text{core}(T) : X/T^{-1}(0) \longrightarrow \text{cl}\, T(X)$ defined by setting

$$\text{core}(T)(x + T^{-1}(0)) = Tx \in \text{cl}\, T(X) \quad \text{for each } x \in X$$

is always one-one and dense; when it happens to be invertible the operator T is called *proper* ([2] Definition 3.2.7). The operator $T \in \mathcal{L}(X, Y)$ is called *regular* if there is $T' \in \mathcal{L}(Y, X)$ for which

$$T = TT'T.$$

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It is known ([2] Theorem 3.8.2) that $T \in \mathcal{L}(X,Y)$ is regular if and only if T is proper and both $T^{-1}(0)$ and $T(X)$ are complemented. Evidently

$$(0.1) \qquad T \text{ regular} \Longrightarrow T \text{ proper} \Longrightarrow T \text{ relatively open}$$

Relative openness can be tested with the (*reduced*) *minimum modulus*

$$\gamma(T) = \inf\{\|Tx\| : \operatorname{dist}(x, T^{-1}(0)) \geq 1\} \quad \text{if } 0 \neq T \in \mathcal{L}(X,Y),$$

if $T = 0$ we may take $\gamma(T) = \infty$. Evidently

$$T \text{ relatively open} \Longleftrightarrow \gamma(T) > 0.$$

To prove the main result we need to two lemmas.

LEMMA 1. *If $T \in \mathcal{L}(X,Y)$ is relatively open and if M is a subspace of X then the restriction of T to $M + T^{-1}(0)$ is relatively open.*

Proof. If T_1 is the restriction of T to $M + T^{-1}(0)$ then $T_1^{-1}(0) = T^{-1}(0)$, and hence $0 < \gamma(T) \leq \gamma(T_1)$, which says that T_1 is relatively open.

LEMMA 2. *Suppose X is a normed space and $T \in \mathcal{L}(X)$. If A, B and D are closed subspaces of X for which*

$$T(A) \cap D = \{0\} \quad \textit{and} \quad \dim B = n < \infty$$

then $\dim(T(A+B) \cap D) \leq n$.

Proof. Let $T(A) \cap D = \{0\}$ and $\dim B = n < \infty$. Write

$$W = T(A+B) \cap D.$$

Suppose that $\{e_i : i \in \Gamma\}$ for some set Γ is a subset of W containing a (algebraic) basis for W. Then there are sets $\{a_i : i \in \Gamma\} \subseteq A$ and $\{b_i : i \in \Gamma\} \subseteq B$ for which

$$(2.1) \qquad e_i = T(a_i + b_i) \quad \text{for each } i \in \Gamma.$$

Suppose that $\{b_{j_1}, \cdots, b_{j_k}\}$ is a basis for the span of the set $\{b_i : i \in \Gamma\}$, where $j_1, \cdots, j_k$ are in Γ; by the hypothesis on B we have $k \leq n$. Now each $b_i (i \in \Gamma)$ has a unique representation of the form

$$b_i = \alpha_{i_1} b_{j_1} + \cdots + \alpha_{i_k} b_{j_k} \quad \text{for some scalars } \alpha_{i_1}, \cdots, \alpha_{i_k};$$

we thus have

$$(2.2) \qquad e_i = T(a_i) + \alpha_{i_1} T(b_{j_1}) + \cdots + \alpha_{i_k} T(b_{j_k}) \quad \text{for each } i \in \Gamma.$$

From (2.1) we also have

(2.3)

$$\alpha_{i_1} e_{j_1} + \cdots + \alpha_{i_k} e_{j_k} = T(\alpha_{i_1} a_{j_1} + \cdots + \alpha_{i_k} a_{j_k}) + \alpha_{i_1} T(b_{j_1}) + \cdots + \alpha_{i_k} T(b_{j_k}).$$

Thus, by (2.2) and (2.3), we have

$$(2.4) \qquad \alpha_{i_1} e_{j_1} + \cdots + \alpha_{i_k} e_{j_k} - e_i = T(\alpha_{i_1} a_{j_1} + \cdots + \alpha_{i_k} a_{j_k} - a_i).$$

Observe that the left-hand side of (2.4) is in D. Since, moreover, $T(A) \cap D = \{0\}$ it follows that

$$e_i = \alpha_{i_1} e_{j_1} + \cdots + \alpha_{i_k} e_{j_k} \quad \text{for each } i \in \Gamma,$$

which says that $\{e_{j_1}, \cdots, e_{j_k}\}$ is a subset of W containing a basis for W. We can therefore conclude that $\dim W \leq k \leq n$.

We are now ready to prove:

THEOREM 3. *Suppose $T \in \mathcal{L}(X)$ is regular on a normed space X, with finite dimensional intersection $T^{-1}(0) \cap T(X)$. Then there exists $\epsilon > 0$ such that if $S \in \mathcal{L}(X)$ is invertible, commutes with T and $\|S\| < \epsilon$ then the null space of $T - S$ is finite dimensional.*

Proof. Suppose $T = TT'T \in \mathcal{L}(X)$ is regular and $T^{-1}(0) \cap T(X)$ is finite dimensional. Then $T'T(X)$ is the complementary subspace to $T^{-1}(0)$: that is, $X = T'T(X) \oplus T^{-1}(0)$. If T_1 is the restriction of T to $T'T(X)$ then T_1 is bounded below. Thus there exists $\epsilon > 0$ such that if $S \in \mathcal{L}(X)$ is invertible with $ST = TS$ and $\|S\| < \epsilon$ and if S_1 is the restriction of S to $T'T(X)$ then $T_1 - S_1$ is also bounded below because the set of all bounded below operators forms an open set (cf. [1] Theorem V.1.6; [2] Theorem 3.3.3). Since $T_1^{-1}(T^{-1}(0) \cap T(X))$ is finite dimensional, we can find a closed subspace H of $T'T(X)$ for which

$$H \oplus T_1^{-1}(T^{-1}(0) \cap T(X)) = T'T(X). \tag{3.1}$$

If T_2 is the restriction of T to H then T_2 is also bounded below. If T_3 is the restriction of T to $T(X) + T^{-1}(0)$ then, by Lemma 1, T_3 is relatively open. In particular, since by (3.1)

$$T(H) \oplus (T^{-1}(0) \cap T(X)) = T(X),$$

we have

$$T(H) \oplus T^{-1}(0) = T(X) + T^{-1}(0);$$

thus if T_4 is the restriction of T to $T(H)$ then T_4 is bounded below. Further, the product T_4T_2 is well defined and bounded below ([2] Theorem 3.3.2). We now claim that

$$(T_1 - S_1)(H) \cap T^{-1}(0) = \{0\}. \tag{3.2}$$

Indeed, if this is not so then we can find a sequence (x_n) in H with $\|x_n\| = 1$ and a sequence $(S^{(n)})$ in $\mathcal{L}(X)$ with $\|S^{(n)}\| \longrightarrow 0$ as $n \longrightarrow \infty$ for which

$$(T_1 - S_1^{(n)})(x_n) \in T^{-1}(0),$$

where $S_1^{(n)}$ is the restriction of $S^{(n)}$ to $T'T(X)$, so that

$$T_4T_2(x_n) = TT_1(x_n) = TS_1^{(n)}(x_n) \longrightarrow 0,$$

which contradicts the fact that T_4T_2 is bounded below. Therefore it follows from (3.1) and (3.2) that

$$\begin{aligned} M &= (T_1 - S_1)(T'TX) \cap T^{-1}(0) \\ &= \{(T_1 - S_1)(H \oplus T_1^{-1}(T^{-1}(0) \cap TX))\} \ \cap \ T^{-1}(0) \end{aligned}$$

is finite dimensional, because the conditions of Lemma 2 are satisfied with $T_1 - S_1$ in place of T and $H = A$, $T_1^{-1}(T^{-1}(0) \cap T(X)) = B$, $T^{-1}(0) = D$. It therefore follows from the fact that $\dim M < \infty$ and $ST^{-1}(0) = T^{-1}(0)$ that, for each $x = y + z \in X$ with $y \in T'TX$ and $z \in T^{-1}(0)$ we have

$$\begin{aligned} x \in (T-S)^{-1}(0) &\Longrightarrow (T_1 - S_1)(y) - S(z) = 0 \\ &\Longrightarrow (T_1 - S_1)(y) \in M \text{ and } S(z) \in M \\ &\Longrightarrow x \in F + G \text{ with } F = (T_1 - S_1)^{-1}(M) \\ &\qquad\qquad\qquad \text{and } G = S^{-1}(M), \end{aligned}$$

where $F + G$ must be finite dimensional; thus $(T - S)^{-1}(0)$ is finite dimensional.

For brevity, we shall write

$$\sigma^p_{ess}(T) = \{\lambda \in C : (T - \lambda I)^{-1}(0) \text{ is infinite dimensional}\}$$

for the *essential point spectrum* of $T \in \mathcal{L}(X)$.

We conclude with:

THEOREM 4. *If $T \in \mathcal{L}(X)$ is regular with finite dimensional intersection $T^{-1}(0) \cap T(X)$ then we have*

$$0 \notin acc\, \partial \sigma^p_{ess}(T), \tag{4.1}$$

where $acc\, K$ denotes the accumulation points of $K \subseteq C$.

Proof. Applying Theorem 3 to $T - S$ with $S = \lambda I$ and $0 < |\lambda| < \delta$ for some δ gives that the dimension of $(T - \lambda I)^{-1}(0)$ is finite on a punctured neighborhood of 0.

In fact, (4.1) says that

$$0 \in \partial\, \sigma^p_{ess}(T) \Longrightarrow 0 \in \operatorname{iso} \sigma^p_{ess}(T),$$

where $\operatorname{iso} K$ denotes the isolated points of $K \subseteq C$.

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DEPARTMENT OF MATHEMATICS, SUNG KYUN KWAN UNIVERSITY, SUWON 440–746, KOREA