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CERTAIN EXACT COMPLEXES ASSOCIATED TO THE PIERI TYPE SKEW YOUNG DIAGRAMS

YOO BONG CHUN AND HYOUNG J. KO

1. Introduction

The characteristic free representation theory of the general linear group has found a wide range of applications, ranging from the theory of free resolutions to the symmetric function theory. Representation theory is used to facilitate the calculation of explicit free resolutions of large classes of ideals (and modules). Recently, K. Akin and D. A. Buchsbaum [2] realized the Jacobi-Trudi identity for a Schur function as a resolution of GL_n-modules. Over a field of characteristic zero, it was observed by A. Lascoux [6]. T. Józefiak and J. Weyman [5] used the Koszul complex to realize a formula of D. E. Littlewood as a resolution of Schur modules. This leads us to further study resolutions of Schur modules of a particular form.

In this article we will describe some new classes of finite free resolutions associated to the Pieri type skew Young diagrams. As a special case of these finite free resolutions we obtain the generalized Koszul complex constructed in [1].

In section 2 we review some of the basic definitions and properties of Schur modules that we shall use.

In section 3 we describe certain exact complexes associated to the Pieri type skew partitions.

Throughout this article, unless otherwise specified, R is a commutative ring with an identity element and a mudule F is a finitely generated free R-module.

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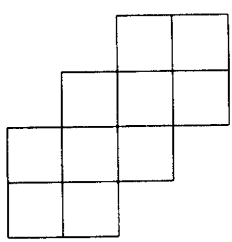
2. Schur modules

In this section we review some of the basic facts in the characteristic free representation theory of GL_n. For detailed definitions of the terms and complete proofs of the propositions, we refer to [2] and [3].

A *partition* is any finite sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ of nonnegative integers in nonincreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. The *length* of the partition λ is the number of positive terms in the sequence, and the *weight* of a partition λ is the sum of the terms of λ. It is often convenient not to distinguish between $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ and $(\lambda_1, \lambda_2, \cdots, \lambda_r, 0)$. If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ is a partition, we sometimes write $\lambda = (h_1^{f_1}, h_2^{f_2}, \cdots)$ where exactly f_i of the λ_j are equal to h_i.

The partition μ is *contained* in the patition $\lambda (\mu \subseteq \lambda)$ if $\mu_i \leq \lambda_i$ for all i. Given a pair (λ, μ) of partitions such that $\mu \subseteq \lambda$, a relative sequence $(\lambda_1 - \mu_1, \lambda_2 - \mu_2, \cdots)$ is called a *skew partition* and is denoted by λ/μ. Observe that any partition λ can be regarded as a skew partition $\lambda/(0, 0, \cdots)$.

The *diagram*, or *shape*, of a skew partition $\lambda/\mu = (\lambda_1, \cdots, \lambda_r)/(\mu_1, \cdots, \mu_r)$, denoted by $\nabla_{\lambda/\mu}$, may be formally defined as the set of points $(i, j) \in Z \times Z$ such that $\mu_i + 1 \leq j \leq \lambda_i$ for $i = 1, \cdots, r$. We use the ordering of the pairs (i, j) that we use for matrices. That is, the i's increase downward and the j's increase from left to right. Thus, if $\lambda = (4, 4, 3, 2)$ and $\mu = (2, 1, 0)$, then $\nabla_{\lambda/\mu}$ is usually depicted as



where each box represents an ordered pair (i, j).

DEFINITION 2.1. Let F be a finitely generated free R-module and k a nonnegative integer. We denote the k^{th} exterior and symmetric modules of F by $\Lambda^k F$ and $S_k F$, respectively. Further, we define $\Lambda F = \bigoplus_{k \geq 0} \Lambda^k F$ and $SF = \bigoplus_{k \geq 0} S_k F$. Then they become Hopf algebras with multiplications m and the comultiplications Δ. (m is defined as usual, and Δ is induced by the diagonalization. Note that $\Lambda^k F$ and $S_k F$ are ploynomial $GL(F)$-modules ; moreover m and Δ are morphisms of $GL(F)$-modules.)

Suppose now that $A = (a_{ij})$ is an $r \times p$ shape matrix, that is, all of its entries are either 0 or 1. For each $i = 1, \cdots, r$, let $\mathrm{a}_i = \sum_{j=1}^{p} a_{ij}$ and for each $j = 1, \cdots, p$, let $b_j = \sum_{i=1}^{r} a_{ij}$. Then for a free R-module F we cah define the map

$$d_A(F) : \Lambda^{\mathrm{a}_1} F \otimes \cdots \otimes \Lambda^{\mathrm{a}_r} F \longrightarrow S_{b_1} F \otimes \cdots \otimes S_{b_p} F$$

to be the composite

$$\begin{aligned}
&\Lambda^{\mathrm{a}_1} F \otimes \cdots \otimes \Lambda^{\mathrm{a}_r} F \\
&\longrightarrow (\Lambda^{a_{11}} F \otimes \cdots \otimes \Lambda^{a_{1p}} F) \otimes \cdots \otimes (\Lambda^{a_{r1}} F \otimes \cdots \otimes \Lambda^{a_{rp}} F) \\
&\cong (\Lambda^{a_{11}} F \otimes \cdots \otimes \Lambda^{a_{r1}} F) \otimes \cdots \otimes (\Lambda^{a_{1p}} F \otimes \cdots \otimes \Lambda^{a_{rp}} F) \\
&\cong (S_{a_{11}} F \otimes \cdots \otimes S_{a_{r1}} F) \otimes \cdots \otimes (S_{a_{1p}} F \otimes \cdots \otimes S_{a_{rp}} F) \\
&\longrightarrow S_{b_1} F \otimes \cdots \otimes S_{b_p} F
\end{aligned}$$

where the first map is comultiplication, the second is the isomorphism rearranging terms, the third is the isomorphism identifying $\Lambda^{a_{ij}} F$ with $S_{a_{ij}} F$(for $a_{ij} = 0$ or 1), and the last map is multiplication. The map $d_A(F)$ is called the *Schur map* on F associated to the matrix A.

Now let λ/μ be a skew partition, and let $A_{\lambda/\mu}$ be the $r \times \lambda_1$ shape matrix whose entries a_{ij} are defined as follows: $a_{ij} = 1$ if $\mu_i + 1 \leq j \leq \lambda_i$, and $a_{ij} = 0$ otherwise. Then the map $d_{A_{\lambda/\mu}}(F)$ is called the *Schur map* on F of shape λ/μ. (One usually write $d_{\lambda/\mu}(F)$ in place of $d_{A_{\lambda/\mu}}(F)$.)

DEFINITION 2.2. The image of $d_A(F)$, denoted by $L_A F$, is called the Schur mudule on F associated to the shape matrix A. When A is the shape matrix of a skew partition λ/μ, we write $L_{\lambda/\mu} F$ instead of $L_A F$. (When R is a field of characteristic zero, $L_\lambda F$ is an irreducible homogenious polynomial $GL(F)$-module of degree $|\lambda|$ corresponding to the partition λ.)

Notice that if $\mu = (0)$, then $\nabla_{\lambda/\mu} = \nabla_\lambda$ and we may write $L_\lambda F$ and $d_\lambda(F)$ for $L_{\lambda/\mu} F$ and $d_{\lambda/\mu}(F)$. Also notice that if λ is a partition, and if we denote by $\tilde{\lambda}_j$ the number of boxes in the j^{th} column of ∇_λ, then the sequence $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots)$ is also a partition, called the *transpose* of the partition λ. Thus, letting $A_{\lambda/\mu}$ denote the matrix described above, we see that the integers b_j associated to $A_{\lambda/\mu}$ are simply $\tilde{\lambda}_j - \tilde{\mu}_j$, so that the map $d_{\lambda/\mu}(F)$ is a map from $\Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_r-\mu_r}F$ to $S_{\tilde{\lambda}_1-\tilde{\mu}_1}F \otimes \cdots \otimes S_{\tilde{\lambda}_p-\tilde{\mu}_p}F$ where $p = \lambda_1$. This is usually abbreviated to $d_{\lambda/\mu}(F) : \Lambda_{\lambda/\mu}F \longrightarrow S_{\tilde{\lambda}/\tilde{\mu}}F$.

Two shape matrices A and B are said to be *equivalent* if one can be transformed into the other by permutations of its rows and columns. It is proved in [2] that if A and B are equivalent then the Schur modules L_AF and L_BF are naturally isomorphic.

PROPOSITION 2.3 [3, THEOREM II.2.16]. *For any R, F and λ/μ, $L_{\lambda/\mu}F$ is a universally free R-module.*

DEFINITION 2.4. When p_1, p_2 and k are positive integers such that $k \leq p_2$, we have the $GL(F)$-morphisms

$$\Lambda^{p_1+k}F \otimes \Lambda^{p_2-k}F \xrightarrow{\Delta\otimes 1} \Lambda^{p_1}F \otimes \Lambda^k F \otimes \Lambda^{p_2-k}F \xrightarrow{1\otimes m} \Lambda^{p_1}F \otimes \Lambda^{p_2}F.$$

This composite map will be denoted by $\square_k(F)$ or $\square_k$.

Similarly when $\mathrm{a} = (\mathrm{a}_1, \cdots, \mathrm{a}_r)$ is a sequence of positive integers, we define a $GL(F)$-morphism as

$$\begin{array}{c} \sum_{i=1}^{r-1}\sum_{k=1}^{\mathrm{a}_{i+1}} \Lambda^{\mathrm{a}_1}F \otimes \cdots \otimes \Lambda^{\mathrm{a}_i+k}F \otimes \Lambda^{\mathrm{a}_{i+1}-k}F \otimes \cdots \otimes \Lambda^{\mathrm{a}_r}F \\ \Big\downarrow {\scriptstyle \sum_{i=1}^{r-1}\sum_{k=1}^{\mathrm{a}_{i+1}} 1\otimes\cdots\otimes 1\otimes\square_k\otimes 1\otimes\cdots\otimes 1} \\ \Lambda_{\mathrm{a}}F = \Lambda^{\mathrm{a}_1}F \otimes \cdots \otimes \Lambda^{\mathrm{a}_r}F \end{array}$$

and denote it by $\square_{\mathrm{a}}(F)$ or $\square_{\mathrm{a}}$.

PROPOSITION 2.5 [3, THEOREM II.2.16]. *For any skew partition λ/μ, the following sequence of $GL(F)$-morphisms is exact:*

$$0 \longrightarrow Im(\square_{\lambda/\mu}(F)) \longrightarrow \Lambda_{\lambda/\mu}F \xrightarrow{d_{\lambda/\mu}(F)} L_{\lambda/\mu}F \longrightarrow 0.$$

DEFINITION 2.6. Let $\nabla_{\lambda/\mu}$ be the diagram associated to the skew partition $\lambda/\mu = (\lambda_1, \cdots, \lambda_r)/(\mu_1, \cdots, \mu_r)$ and n a positive integer. Then a tableau of shape λ/μ with values in the set $\{1, 2, \cdots, n\}$ is a function from $\nabla_{\lambda/\mu}$ to $\{1, 2, \cdots, n\}$. The set of all such tableaux is denoted by $Tab_{\lambda/\mu}\{1, 2, \cdots, n\}$. Further, a tableau T is said to be standard if the following two conditions are satisfied:

(I) $T(i,j) < T(i,j+1)$ whenever (i,j) and $(i,j+1)$ are both in $\nabla_{\lambda/\mu}$.

(II) $T(i,j) \leq T(i+1,j)$ whenever (i,j) and $(i+1,j)$ are both in $\nabla_{\lambda/\mu}$.

Moreover let $\{e_1, \cdots, e_n\}$ be a free basis of F. For a tableau T contained in $Tab_{\lambda/\mu}\{1, \cdots, n\}, e_T$ is defined to be an element of $\Lambda_{\lambda/\mu}F$ as follows:

$$e_T = e_{T(1,\mu_1+1)} \wedge \cdots \wedge e_{T(1,\lambda_1)} \otimes \cdots \otimes e_{T(r,\mu_r-1)} \wedge \cdots \wedge e_{T(r,\lambda_r)}.$$

PROPOSITION 2.7 [3, THEOREM II.2.16]. *For any R, F and λ/μ, the following set becomes an R-basis of $L_{\lambda/\mu}F$:*

$$\{d_{\lambda/\mu}(F)(e_T) | T \text{ is a standard tableau contained in } Tab_{\lambda/\mu}\{1, \cdots, n\}\}.$$

Now let $\alpha = (\lambda_1, \lambda_2)/(\mu_1, \mu_2)$ and $\beta = (\lambda_1+1, \lambda_2)/(\mu_1+1, \mu_2)$ be the two-rowed skew partitions. Observing that $\alpha_i = \lambda_i - \mu_i = \beta_i (i = 1, 2)$ we have the equalities $\Lambda^{\alpha_1}F \otimes \Lambda^{\alpha_2}F = \Lambda^{\lambda_1-\mu_1}F \otimes \Lambda^{\lambda_2-\mu_2}F = \Lambda^{\beta_1}F \otimes \Lambda^{\beta_2}F$. Then the identity map $\Lambda^{\beta_1}F \otimes \Lambda^{\beta_2}F \longrightarrow \Lambda^{\alpha_1}F \otimes \Lambda^{\alpha_2}F$ on the generators induces a natural surjection $v : L_\beta F \longrightarrow L_\alpha F$. Next we recall from [2] that the kernel of v is the image of the composite map

$$\Lambda^{\lambda_1-\mu_2+1}F \otimes \Lambda^{\lambda_2-\mu_1-1}F \xrightarrow{\square_{\mu_1-\mu_2+1}} \Lambda^{\beta_1}F \otimes \Lambda^{\beta_2}F \xrightarrow{d_\beta(F)} L_\beta F$$

and the image of the composite is exactly the Schur module $L_\gamma F$, where $\gamma = (\lambda_1 - \mu_2 + 1, \lambda_2 - \mu_1 - 1)$. In fact, it is proved as a part of the existence of the fundamental exact sequenece of Schur modules in [2] that there is a short exact sequence:

$$0 \longrightarrow L_\gamma F \xrightarrow{u} L_\beta F \xrightarrow{v} L_\alpha F \longrightarrow 0.$$

In general, we have

PROPOSITION 2.8 [2]. *Let $\alpha = (\lambda_1, \cdots, \lambda_r)/(\mu_1, \cdots, \mu_r)$ be a skew partition. Then*

(a) *If $r = 2$, then there exists a short exact sequence of Schur modules*

$$0 \longrightarrow L_{(\lambda_1-\mu_2+1, \lambda_2-\mu_1-1)}F \xrightarrow{u} L_{(\lambda_1+1,\lambda_2)/(\mu_1+1,\mu_2)}F \xrightarrow{v} L_\alpha F \longrightarrow 0$$

(b) *If $r > 2$, then there exists a short exact sequence of Schur modules*

$$0 \longrightarrow L_\gamma F \xrightarrow{u} L_\beta F \xrightarrow{v} L_\alpha F \longrightarrow 0$$

where $\beta = (\lambda_1, \cdots, \lambda_{r-1}, \lambda_r - 1)/(\mu_1, \cdots, \mu_{r-1}, \mu_r - 1)$ and $\gamma = (\lambda_1, \cdots, \lambda_{r-1}, \lambda_r - 1)/(\mu_1, \cdots, \mu_{r-2}, \mu_r - 1, \mu_{r-1})$.

3. Certain exact complexes

In this section we describe new classes of finite free resolutions of certain Schur modules which are related to the Pieri type skew partitions $\lambda/(h)$ and $\lambda/(1^h)$.

For a free R-module F, we have the complex

$$\cdots \longrightarrow S_{q-1}F \otimes \Lambda^{p+1}F \xrightarrow{\delta} S_qF \otimes \Lambda^p F \xrightarrow{\delta} S_{q+1}F \otimes \Lambda^{p-1}F \longrightarrow \cdots$$

which may be regarded as one of the strands of the Koszul complex associated to the ideal $(x_1, \cdots, x_n)$ in $R[x_1, \cdots, x_n]$, where $n =$ rnak F. Recall that the boundary map $\delta : S_qF \otimes \Lambda^p F \longrightarrow S_{q+1}F \otimes \Lambda^{p-1}F$ is defined as the composite

$$S_qF \otimes \Lambda^p F \xrightarrow{1\otimes\Delta} S_qF \otimes F \otimes \Lambda^{p-1}F \xrightarrow{m\otimes 1} S_{q+1}F \otimes \Lambda^{p-1}F$$

and the image of the map δ is just $L_{(p,1^q)}F$. Thus, we may regard the Koszul complex as a "resolution" of $L_{(p,1^q)}F$:

$$\begin{aligned} 0 \longrightarrow S_{p+q-n}F \otimes \Lambda^n F \longrightarrow \cdots \longrightarrow S_{q-1}F \otimes \Lambda^{p+1}F \\ \longrightarrow S_qF \otimes \Lambda^p F \longrightarrow L_{(p,1^q)}F \longrightarrow 0. \end{aligned}$$

Now let $\alpha = \lambda/(h)$ be Pieri type skew partition, and $S_qF \otimes L_\alpha F \longrightarrow L_{(\lambda,1^q)/(h)}F$ the natural surjection (to be defined shortly). Then we construct some exact complex

$$\begin{aligned}\cdots \longrightarrow S_{q-3}F \otimes X_3 \longrightarrow S_{q-2}F \otimes X_2 \longrightarrow S_{q-1}F \otimes X_1 \\ \longrightarrow S_qF \otimes L_\alpha F \longrightarrow L_{(\lambda,1^q)/(h)}F \longrightarrow 0\end{aligned}$$

where X_i depends on α in such a way that, when $L_\alpha F = \Lambda^p F$, we have $X_i = \Lambda^{p+i}F$.

To facilitate matters, we introduce some notation. If $\lambda = (\lambda_1, \cdots, \lambda_r)$ is a partition and l is a nonnegative integer, we shall denote by $\lambda + l$ the partition $(\lambda_1 + l, \cdots, \lambda_r + l)$. It should be noticed that $\lambda + l$ is denoted by the skew partition $(\lambda_1 + l, \cdots, \lambda_r + l)/(l^{r-1})$ in [1].

LEMMA 3.1. *Let $\lambda/(h) = (\lambda_1, \cdots, \lambda_r)/(h)$ be a Pieri type skew partition with $h \leq \lambda_1$, and $\lambda_1 = p$. Then the Schur map*

$$\begin{aligned}d_{(\lambda,1^q)/(h)}(F) : \Lambda^{\lambda_1-h}F \otimes \Lambda^{\lambda_2}F \otimes \cdots \otimes \Lambda^{\lambda_r}F \otimes \underbrace{F \otimes \cdots \otimes F}_{q} \\ \longrightarrow S_{\tilde\lambda_1+q-1}F \otimes S_{\tilde\lambda_2-1}F \otimes \cdots \otimes S_{\tilde\lambda_h-1}F \otimes S_{\tilde\lambda_{h+1}}F \otimes \cdots \otimes S_{\tilde\lambda_p}F\end{aligned}$$

is the composite

$$\begin{aligned}&\Lambda^{\lambda_1-h}F \otimes \Lambda^{\lambda_2}F \otimes \cdots \otimes \Lambda^{\lambda_r}F \otimes \underbrace{F \otimes \cdots \otimes F}_{q} \xrightarrow{1\otimes\cdots\otimes 1\otimes m} \Lambda_{\lambda/(h)}F \otimes S_qF \\ &\xrightarrow{d_{\lambda/(h)}(F)\otimes 1} S_{\tilde\lambda/(\tilde h)}F \otimes S_qF \\ &= S_{\tilde\lambda-1}F \otimes \cdots \otimes S_{\tilde\lambda_h-1}F \otimes S_{\tilde\lambda_{h+1}}F \otimes \cdots \otimes S_{\tilde\lambda_p}F \otimes S_qF \\ &\cong S_{\tilde\lambda_1-1}F \otimes S_qF \otimes S_{\tilde\lambda_2-1}F \otimes \cdots \otimes S_{\tilde\lambda_h-1}F \otimes S_{\tilde\lambda_{h+1}}F \otimes \cdots \otimes S_{\tilde\lambda_p}F \\ &\xrightarrow{m\otimes 1\otimes\cdots\otimes 1} S_{\tilde\lambda_1+q-1}F \otimes S_{\tilde\lambda_2-1}F \otimes \cdots \otimes S_{\tilde\lambda_h-1}F \otimes S_{\tilde\lambda_{h+1}}F \otimes \cdots \otimes S_{\tilde\lambda_p}F.\end{aligned}$$

Proof. It follows immediately from the associativity of multiplication m in the Hopf algebra SF.

Since $1 \otimes \cdots \otimes 1 \otimes m$ is surjective, the image of $d_{\lambda/(h)}(F) \otimes 1$ clearly gets mapped surjectively onto the image of $d_{(\lambda,1^q)/(h)}(F)$ and it is this surjection of $S_qF \otimes L_{\lambda/(h)}F$ onto $L_{(\lambda,1^q)/(h)}F$ which generalizes the surjection of $S_qF \otimes \Lambda^p F$ onto $L_{(p,1^q)}F$.

Now let l be a positive integer, and let $\lambda/(h)$ be a Pieri type skew partition of length r. Then the Koszul map

$$\delta : S_kF \otimes \Lambda^{\lambda_r+l}F \longrightarrow S_{k+1}F \otimes \Lambda^{\lambda_r+l-1}F$$

induces the map

$$\begin{aligned}
&S_kF \otimes \Lambda^{\lambda_1-h}F \otimes \Lambda^{\lambda_2}F \otimes \cdots \otimes \Lambda^{\lambda_{r-1}}F \otimes \Lambda^{\lambda_r+l}F\\
\cong\ &\Lambda^{\lambda_1-h}F \otimes \Lambda^{\lambda_2}F \otimes \cdots \otimes \Lambda^{\lambda_{r-1}}F \otimes S_kF \otimes \Lambda^{\lambda_r+l}F\\
\xrightarrow{1\otimes\delta}\ &\Lambda^{\lambda_1-h}F \otimes \Lambda^{\lambda_2}F \otimes \cdots \otimes \Lambda^{\lambda_{r-1}}F \otimes S_{k+1}F \otimes \Lambda^{\lambda_r+l-1}F\\
\cong\ &S_{k+1}F \otimes \Lambda^{\lambda_1-h}F \otimes \Lambda^{\lambda_2}F \otimes \cdots \otimes \Lambda^{\lambda_{r-1}}F \otimes \Lambda^{\lambda_r+l-1}F.
\end{aligned}$$

Then we have a canonical map

$$\partial : S_kF \otimes L_{\lambda+l/(h+l,l^{r-2})}F \longrightarrow S_{k+1}F \otimes L_{\lambda+l-1/(h+l-1,(l-1)^{r-2})}F$$

which is induced, on the generator level, by the map $1 \otimes \delta$. Indeed, if $x \otimes y_1 \otimes \cdots \otimes y_r$ is any basis element of $S_kF \otimes \Lambda_{\lambda+l/(h+l,l^{r-2})}F$ and $\Delta(y_r) = \sum y_{r_i} \otimes y'_{r_i} \in F \otimes \Lambda^{\lambda_r+l-1}F$, then the map ∂ sends $x \otimes d_{\lambda+l/(h+l,l^{r-2})}(F)(y_1 \otimes \cdots \otimes y_r)$ to

$$\sum xy_{r_i} \otimes d_{\lambda+l-1/(h+l-1,(l-1)^{r-2})}(F)(y_1 \otimes \cdots \otimes y_{r-1} \otimes y'_{r_i}).$$

We will now proceed to describe the exact complexes associated to the Pieri type skew partitions. It is clear that for every positive integer l, and skew partitions $\lambda/(h)$ and $\lambda/(1^h)$ of length r, there are natural isomorphisms

$$\begin{aligned}
\Lambda^l F \otimes L_{\lambda/(h)}F &\cong L_{(\lambda,0)+l/(h+l,l^{r-1})}F \quad \text{and}\\
\Lambda^l F \otimes L_{\lambda/(1^h)}F &\cong L_{(\lambda,0)+l/((l+1)^h,l^{r-h})}F
\end{aligned}$$

because their shapes are equivalent. Thus we have the following results.

PROPOSITION 3.2. *Let l be any positive integer and $\lambda = (\lambda_1, \cdots, \lambda_r)$ any partition of length r. Then*

(a) *If $h \leq \lambda_1$ then there is a short exact sequence of Schur modules*

$$0 \longrightarrow L_{\lambda+l/(h+l,l^{r-2})}F \xrightarrow{u} \Lambda^l F \otimes L_{\lambda/(h)}F$$
$$\xrightarrow{v} L_{(\lambda,1)+l-1/(h+l-1,(l-1)^{r-1})}F \longrightarrow 0.$$

(b) *If $h \leq r$, then there is a short exact sequence of Schur modules*

$$0 \longrightarrow L_{\lambda+l/((l+1)^h,l^{r-h-1})}F \xrightarrow{u} \Lambda^l F \otimes L_{\lambda/(1^h)}F$$
$$\xrightarrow{v} L_{(\lambda,1)+l-1/(l^h,(l-1)^{r-h})}F \longrightarrow 0.$$

Proof. These are special cases of Proposition 2.8.

THEOREM 3.3. *Let F be a free R-module, $\lambda/(h)$ any Pieri type skew partition of length r with $h \leq \lambda_1$, and q a positive integer. Then the following sequence is exact:*

$(*)$

$$\cdots \longrightarrow S_{q-2}F \otimes L_{\lambda+2/(h+2,2^{r-2})}F \xrightarrow{\partial} S_{q-1}F \otimes L_{\lambda+1/(h+1,1^{r-2})}F$$
$$\xrightarrow{\partial} S_q F \otimes L_{\lambda/(h)}F \xrightarrow{\partial} L_{(\lambda,1^q)/(h)}F \longrightarrow 0.$$

Proof. Observe that when $r = 1$, $L_{\lambda+l/(h+l)}F = \Lambda^{\lambda_1-h}F$, so that the sequence $(*)$ reduces to the Koszul complex in that case.

The proof proceeds by induction on q. The case $q = 1$ is the special case of Proposition 3.2 in which $l = 1$. Assuming now that the theorem is true for q and the Pieri type skew partition $(\lambda, 1)/(h)$, we consider the map of complexes

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d, dotted] \\
S_{q-1}F\otimes\Lambda^2F\otimes L_{\lambda/(h)}F \arrow[r, "1\otimes v"] \arrow[d, "\delta\otimes 1"] & S_{q-1}F\otimes L_{(\lambda,1)+1/(h+1,1^{r-2})}F \arrow[d, dotted, "\partial"] \\
S_qF\otimes F\otimes L_{\lambda/(h)}F \arrow[r, "1\otimes v"] \arrow[d, "\delta\otimes 1"] & S_qF\otimes L_{(\lambda,1)/(h)}F \arrow[d, "\partial"] \\
S_{q+1}F\otimes L_{\lambda/(h)}F \arrow[r, "\partial"] \arrow[d] & L_{(\lambda,1^{q+1})/(h)}F \arrow[d] \\
0 & 0
\end{tikzcd}$$

Then it is easy to see that this is indeed a commutative diagram. The kernels of the maps $1\otimes v$ are, by Proposition 3.2, the modules $S_{q-1}F\otimes L_{\lambda+l/(h+l,l^{r-2})}F$ for $l\geq 0$. The induction hypothesis on q, and the acyclicity of the Koszul complex, tell us that the two complexes above are exact. Then the simple homological argument completes the proof that $(*)$ is exact for $q+1$.

If we consider the case of $h=0$ in Theorem 3.3, then the above theorem implies the well-known result [1]:

If F is a free R-module, λ is a partition of length r, and q is a positive integer, then

$$\begin{aligned}\cdots\longrightarrow S_{q-2}F\otimes L_{\lambda+2/(2^{r-1})}F\longrightarrow S_{q-1}F\otimes L_{\lambda+1/(1^{r-1})}F\\ \longrightarrow S_qF\otimes L_\lambda F\longrightarrow L_{(\lambda,1^q)}F\longrightarrow 0\end{aligned}$$

is an exact sequence.

Next we describe another class of exact complexes associated to the second Pieri type skew partition $\lambda/(1^h)$ of lengty r.
To do this, we follow the same line of reasoning as that of Theorem 3.3. We start with the canonical map

$$\partial: S_kF\otimes L_{\lambda+l/((l+1)^h,l^{r-h-1})}F\longrightarrow S_{k+1}F\otimes L_{\lambda+l-1/(l^h,(l-1)^{r-h-1})}F.$$

which, in fact, is induced by the Koszul map

$$\delta : S_kF \otimes \Lambda^{\lambda_r+l}F \longrightarrow S_{k+1}F \otimes \Lambda^{\lambda_r+l-1}F.$$

Now using Proposition 3.2(b) and the same argument given in the proof of Theorem 3.3, we obtain

THEOREM 3.4. *Let F be a free R-module, $\lambda/(1^h) = (\lambda_1, \cdots, \lambda_r)/(1, \cdots, 1)$ any Pieri type skew partition with $\lambda_r \neq 0$ and $h \leq r$, and q any positive integer. Then the following sequence is exact:*

$$\cdots \longrightarrow S_{q-2}F \otimes L_{\lambda+2/(3^h,2^{r-h-1})}F \longrightarrow S_{q-1}F \otimes L_{\lambda+1/(2^h,1^{r-h-1})}F \longrightarrow S_qF \otimes L_{\lambda/(1^h)}F \longrightarrow L_{(\lambda,1^q)/(1^h)}F \longrightarrow 0.$$

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DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, SEOUL 120-749, KOREA