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OPERATORS IN $L(X, Y)$ IN WHICH $K(X, Y)$ IS A SEMI M–IDEAL

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1. Introduction

Since Alfsen and Effros [1] introduced the notion of an M-ideal, many authors [3, 6, 9, 12] have worked on the problem of finding those Banach spaces X and Y for which $K(X, Y)$, the space of all compact linear operators from X to Y, is an M-ideal in $L(X, Y)$, the space of all bounded linear operators from X to Y. The M-ideal property of $K(X, Y)$ in $L(X, Y)$ gives some informations on X, Y and $K(X, Y)$. If $K(X)$ $(= K(X, X))$ is an M-ideal in $L(X)$ $(= L(X, X))$, then X has the metric compact approximation property [5] and X is an M-ideal in X^{**} [10]. If X is reflexive and $K(X)$ is an M-ideal in $L(X)$, then $K(X)^{**}$ is isometrically isomorphic to $L(X)$ [5].

A weaker notion is a semi M-ideal. Studies on Banach spaces X and Y for which $K(X, Y)$ is a semi M-ideal in $L(X, Y)$ were done by Lima [9, 10]. Lima proved the following results:

THEOREM 1.1 [10]. *If $K(X)$ is a semi M-ideal in $L(X)$, then the following holds:*

(a) *If (f_α) is a net in X^* such that $f_\alpha \to f$ in weak *-topology and $\|f_\alpha\| = \|f\| = 1$ for all α, then $f_\alpha \to f$ in norm. More generally, if the closed unit ball B_{X^*} of X^* is weak*-dentable, then T^* is weak* to norm continuous on $\{f \in Y^* : \|f\| = 1\}$ for every $T \in L(X, Y)$.*

(b) *If the closed unit ball B_X of X is dentable, then X is reflexive.*

The purpose of this paper is to study, using Lima's idea, behaviors of operators in $L(X, Y)$ in which $K(X, Y)$ is a semi M-ideal. In Section

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3, under the assumption that $K(X,Y)$ is a semi M-ideal in $L(X,Y)$ we will prove the following;

(a′) *If B_Y is dentable and (x_α) is a net in B_X such that $x_\alpha \to x_0$ ($\|x_0\| = 1$) weakly, then $Tx_\alpha \to Tx_0$ in norm for every $T \in L(X,Y)$, and hence every $T \in L(X,Y)$ is weak to norm continuous on $\{x \in x : \|x\| = 1\}$.*

(b′) *If B_Y is dentable and X has the property "δ" for some $\delta > 0$ which will be defined in Section 3, then no operator in $L(X,Y)$ is bounded below, and hence X is not isomorphic to a subspace of Y.*

2. Preliminaries

A closed subspace J of a Banach space X is called an L-summand if there is a projection P on X such that $PX = J$ and $\|x\| = \|Px\| + \|(I-P)x\|$ for every $x \in X$. A closed subspace J of X is called a semi L-summand if for every $x \in X$ there is a unique $y \in J$ such that $\|x - y\| = \inf\{\|x - z\| : z \in J\}$, and moreover this y satisfies $\|x\| = \|y\| + \|x - y\|$. A closed subspace J of X is called an M-ideal (resp. a semi M-ideal) if J^0, the annihilator of J in X^*, is an L-summand (resp. a semi L-summand) in X^*.

Alfsen and Effros [1] characterized M-ideals in terms of intersection properties of open balls without any reference to the dual space. Later Lima [8] gave the following characterizations of M-ideals and semi M-ideals by intersection properties of closed balls.

THEOREM 2.1 [8]. *Let J be a closed subspace of a Banach space X. The following statements are equivalent :*

(i) *J is an M-ideal in X.*

(ii) *J satisfies the n-ball property for every $n \geq 3$. That is, if $\{B(a_i, r_i)\}_{i=1}^n$ is a family of closed balls in X such that*

$$\bigcap_{i=1}^{n} B(a_i, r_i) \neq \emptyset \text{ and } J \cap B(a_i, r_i) \neq \emptyset$$

for each i, then for every $\varepsilon > 0$

$$J \cap \bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) \neq \emptyset,$$

where a_i and r_i are the center and the radius of $B(a_i, r_i)$, respectively.

(iii) *J satisfies the 3-ball property.*

(iv) *Same as (ii) but with $n = 3$ and $a_i = x + x_i$ with $x \in B_X, x_i \in B_J$ for $i = 1, 2, 3$.*

THEOREM 2.2 [8]. *Let J be a closed subspace of a Banach space X. Then the following statements are equivalent:*

(i) *J is a semi M-ideal in X.*

(ii) *J satisfies the 2-ball property.*

(iii) *For any $\varepsilon > 0$, any $x \in B_X$ and any $j \in B_J$, there exists $z \in J$ such that*

$$| x \pm j - z | < 1 + \varepsilon.$$

A slice of a closed bounded convex set C in a Banach space X is a set of the form

$$S(x^*, \alpha) = \{x \in C : x^*(x) + \alpha \geq \sup x^*(C)\}$$

where $x^* \in X^*$, $\|x^*\| = 1$ and $\alpha > 0$.

A closed bounded convex set C is said to be dentable if it has slices of arbitrarily small diameter [4, p.199]. It is known that a Banach space X has the Radon-Nikodym property if and only if each of its equivalent norm has the dentable closed unit ball [4, p.204]. The class of Banach spaces with the Radon-Nikodym property comprises reflexive spaces, separable dual spaces and many others.

3. Results

In this section we prove some properties possessed by all operators in $L(X,Y)$ when $K(X,Y)$ is a semi M-ideal in $L(X,Y)$, and X and Y satisfy additional conditions. The following theorem is an analogue of Theorem 1.1 (a) and the proof given below is a minor modification of Lima's proof of Theorem 1.1 (a).

THEOREM 3.1. *Suppose $K(X,Y)$ is a semi M-ideal in $L(X,Y)$ and B_Y is dentable. If (x_α) is a net in X such that $x_\alpha \to x_0$ weakly and $\|x_\alpha\| \leq \|x_0\| = 1$ for all α, then $Tx_\alpha \to Tx_0$ in norm for every $T \in L(X,Y)$, and hence every T in $L(X,Y)$ is weak to norm continuous on $\{x \in X : \|x\| = 1\}$.*

Proof. Suppose (x_α) is a net in X such that $x_\alpha \to x_0$ weakly and $\|x_\alpha\| \leq \|x_0\| = 1$ for all α. Let $\varepsilon > 0$. Since B_Y is dentable, there exists $g \in Y^*$ with $\|g\| = 1$ and $0 < t < 1$ such that the slice

$$S(g,t) = \{y \in B_Y : g(y) > 1 - t\}$$

has diameter less than ε. By the Bishop-Phelps theorem [4, p.189] we may assume that there exists $y_0 \in Y$ such that $g(y_0) = 1 = \|y_0\|$.

Choose $f \in X^*$ with $\|f\| = 1 = f(x_0)$ and define $S : X \to Y$ by

$$Sx = f(x)y_0 \quad \text{for } x \in X.$$

Then $S \in K(X,Y)$ and $\|S\| = 1$.

Let $T \in L(X,Y)$ with $\|T\| \leq 1$. Since $K(X,Y)$ is a semi M-ideal in $L(X,Y)$, by Theorem 2.2 there exists $U \in K(X,Y)$ such that

$$\|S \pm (T-U)\| \leq 1 + t/4.$$

For $x \in B_X$ with $f(x) > 1 - t/4$, we get

$$\begin{aligned} 1 + t/4 &\geq \|Sx \pm (T-U)x\| \\ &\geq |f(x)g(y_0) \pm g(T-U)x| \\ &> 1 - t/4 \pm g(T-U)x. \end{aligned}$$

Hence $|\ g(T-U)x\ | < 2t/4$ and

$$\begin{aligned} g\left(\frac{f(x)y_0 \pm (T-U)x}{1+t/4}\right) &> \frac{(1-t/4) - 2t/4}{1+t/4} \\ &> 1 - t. \end{aligned}$$

Since $\dfrac{Sx \pm (T-U)x}{1+t/4}$ is in $S(g,t)$ and diam $S(g,t) < \varepsilon$, we get

$$\begin{aligned}\|(T-U)x\| &\le \frac{1}{1+t/4}\,\|\,\{Sx+(T-U)x\} - \{Sx-(T-U)x\}\|\\ &< \varepsilon\end{aligned}$$

for all $x \in B_X$ with $f(x) > 1-t/4$.

Since $f(x_\alpha) \to f(x_0) = 1$ and a compact operator carries a weakly convergent net to a norm convergent net, there exists α_0 such that

$$f(x_\alpha) > 1-t/4 \text{ and } \|Ux_\alpha - Ux_0\| < \varepsilon$$

for $\alpha \ge \alpha_0$. Hence we get that for $\alpha \ge \alpha_0$

$$\begin{aligned}\|Tx_\alpha - Tx_0\| &\le \|(T-U)x_\alpha\| + \|Ux_\alpha - Ux_0\| + \|(T-U)x_0\|\\ &< 3\varepsilon.\end{aligned}$$

Let us say that a Banach space X has the property "δ" for $\delta > 0$ if for each $0 < c < 1$ there exist a sequence (x_n) in B_X and $f \in B_{X^*}$ such that $\|x_n - x_m\| \ge \delta$ and $f(x_n) \ge c$ for all distinct n and m.

Recall that an operator $T \in L(X,Y)$ is said to be bounded below if there exists $M > 0$ such that $\|Tx\| \ge M\|x\|$ for all $x \in X$, equivalently T is an isomorphism between X and $T(X)$.

THEOREM 3.2. *Suppose X and Y are Banach spaces and $K(X,Y)$ is a semi M-ideal in $L(X,Y)$. If X has the property "δ" for some $\delta > 0$ and B_Y is dentable, then no operator in $L(X,Y)$ is bounded below, and hence X is not isomorphic to a subspace of Y.*

Proof. For a contradiction suppose there exist $T \in L(X,Y)$ and $M > 0$ such that $\|Tx\| \ge M\|x\|$ for all $x \in X$. We may assume that $\|T\| \le 1$. Choose $\varepsilon > 0$ so that $4\varepsilon < M\delta$. As in Theorem 3.1, let $S(g,t)$ be a slice with diameter less than ε. Again we may assume that $\|g\| = g(z) = 1 = \|z\|$ for some $z \in Y$.

Since X has the property "δ" for some $\delta > 0$, there exist a sequence (x_n) in B_X and $f \in B_{X^*}$ such that $\|x_n - x_m\| \ge \delta$ and $f(x_n) \ge 1-t/4$ for all distinct n and m. We define $S : X \to Y$ by

$$Sx = f(x)z.$$

Then $S \in K(X, Y)$ and $\|S\| \leq 1$.

Since $K(X, Y)$ is a semi M-ideal in $L(X, Y)$, there exists U in $K(X, Y)$ such that

$$\|S \pm (T - U)\| \leq 1 + t/4.$$

Since $f(x_n) > 1 - t/4$ for all n, repeating the same argument used in the proof of Theorem 3.1 we get that

$$\|Tx_n - Ux_n\| \leq \varepsilon$$

for all n. By compactness of U, we have

$$\begin{aligned} M\delta &\leq M\|x_n - x_m\| \\ &\leq \|T(x_n - x_m)\| \\ &\leq \|Tx_n - Ux_n\| + \|Ux_n - Ux_m\| + \|Tx_m - Ux_m\| \\ &\leq 3\varepsilon \end{aligned}$$

for infinitely many n and m. This contradicts to the choice of ε and hence the proof is complete.

By a result of James [7], if X is a non-reflexive Banach space then for each $0 < c < 1$ there exist sequences (x_n) in B_X and (f_n) in B_{X^*} such that

$$\begin{aligned} &\text{(i)}\ \ f_m(x_n) = c \qquad \text{for}\ \ n \geq m \\ &\text{(ii)}\ \ f_m(x_n) = 0 \qquad \text{for}\ \ n < m. \end{aligned}$$

Since $\|x_m - x_n\| \geq f_m(x_m - x_n) = c$ for $n < m$, X has the property "δ" for all $0 < \delta < 1$. Thus from Theorem 3.2 we have the following result of Lima[Theorem 1.1 (b)].

COROLLARY 3.3. *If X is a non-reflexive Banach space and $K(X)$ is a semi M-ideal in $L(X)$, then B_Y is not dentable.*

COROLLARY 3.4. *Suppose Y is a reflexive Banach space and Z is either an infinite dimensional Hilbert space or l_p for $1 < p < \infty$. Let $X = Y \oplus_\infty Z$. Then $K(X)$ is not a semi M-ideal in $L(X)$.*

Proof. Since X is reflexive, B_X is dentable. By Theorem 3.2, it suffices to prove that X has the property "δ" for some $\delta > 0$. Suppose $Z = l_p$. Fix $e \in Y$ with $\|e\| = 1$ and choose $f \varepsilon Y^*$ with $f(e) = \|f\| = 1$. Let (e_n) be the unit vector basis for l_p. Set $x_n = e + e_n \in X$ and regard f as a functional on X. Then $\|x_n\| = 1$, $f(x_n) = f(e) = 1 = \|f\|$ and $\|x_n - x_m\| = 2^{1/p}$ for all distinct n and m. Thus $Y \oplus_\infty l_p$ has the property "$2^{1/p}$".

If Z is an infinite dimensional Hilbert space, we simply replace the unit vector basis (e_n) of l_p by any orthornomal sequence in Z in the above proof.

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