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# INVARIANCE OF THE SPACE OF THETA–SERIES UNDER THETA OPERATORS

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## 0. Introduction and Notations

One of the most powerful methods of studying representations of quadratic forms by forms is via theta-series. Many authors did a great deal of work in this direction after Siegel's pioneering work[Si1]. Unfortunately, however, most of them worked in the case when the representing quadratic form has an even number of variables. The reason is simple : quadratic forms with even number of variables are associated to integral weight theta-series while those with odd number of variables to half integral weight theta-series whose transformation formulas involve branch problems.

In this article, we study the behavior of half integral weight theta-series under theta operators. Theta operators are very important in the study of theta-series in connection with Hecke operators. Andrianov[A1] proved that the space of integral weight theta-series is invariant under the action of theta operators. We prove that his statement can be extened for half integral weight theta-series with a slight modification. By using this result one can prove that the space of theta-series is invariant under the action of Hecke operators as Andrianov did for intrgral weight theta-series[A1].

For an $m \times m$ matrix $g$ and an $m \times n$ matrix $h$, let $g[h] = {}^t hgh$, where ${}^t h$ is the transpose of $h$. For a $2n \times 2n$ matrix $g$, let $A_g, B_g, C_g$, and $D_g$ be the $n \times n$ block matrices in the upper left, upper right, lower left, and lower right corners of $g$, respectively. Let $\mathrm{diag}(N_1, N_2, \ldots, N_r)$ be the matrix with block matrices $N_1, N_2, \ldots, N_r$ on its main diagonal and

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zeroes outside. Let $\mathcal{N}_m$ be the set of all semi-positive definite (eigenvalues $\geq 0$), semi-integral (diagonal entries and twice of nondiagonal entries are integers), symmetric $m \times m$ matrices, and $\mathcal{N}_m^+$ be its subset consisting of positive definite (eigenvalues $> 0$) matrices.

Let

$$G_n = GSp_n^+(\mathbf{R}) = \{g \in M_{2n}(\mathbf{R})\ ;\ J_n[g] = rJ_n,\ r > 0\}$$
$$\Gamma^n = Sp_n(\mathbf{Z}) = \{M \in M_{2n}(\mathbf{Z})\ ;\ J_n[M] = J_n\},$$

where $J_n = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ and $r = r(g) \in \mathbf{R}$ is determined by $g$. And let

$$\mathcal{H}_n = \{Z = X + iY \in M_n(\mathbf{C})\ ;\ {}^tZ = Z, Y > 0\}$$

where $Y > 0$ means $Y$ is positive definite. For $g \in G_n$ and $Z \in \mathcal{H}_n$, we set

$$g\langle Z\rangle = (A_gZ + B_g)(C_gZ + D_g)^{-1} \in \mathcal{H}_n.$$

For $g \in M_n(\mathbf{C})$, $e(g) = \exp(2\pi i\sigma(g))$ where $\sigma(g)$ is the trace of $g$. Finally, we let $\langle n\rangle = n(n+1)/2$ for $n \in \mathbf{Z}$.

For other standard terminologies and basic facts, we refer the readers [A2], [M], [O].

## 1. Siegel Modular Forms

Let $n, q$ be positive integers and $p$ be a prime relatively prime to $q$. Let

$$\Gamma_0^n(q) = \{M \in \Gamma^n\ ;\ C_M \equiv 0 \pmod q\}$$
$$\Gamma_0^n = \{M \in \Gamma^n\ ;\ C_M = 0\}$$
$$L^n = L_p^n = \{g \in M_{2n}(\mathbf{Z}[p^{-1}])\ ;\ J_n[g] = p^\delta J_n, \delta \in \mathbf{Z}\}$$
$$L_0^n = L_{0,p}^n = \{g \in L^n\ ;\ C_g = 0\}$$
$$E^n = E_p^n = \{g \in L^n\ ;\ \delta(g) \in 2\mathbf{Z}\}$$
$$E_0^n = E_{0,p}^n = E^n \cap L_0^n$$

where $\delta = \delta(g) \in \mathbf{Z}$ is determined by $g$. Let

$$\hat{G}_n = \{(g, \alpha(Z))\ ;\ g \in G_n\}$$

where $\alpha(Z)$ is a holomorphic map on $\mathcal{H}_n$ satisfying $\alpha(Z)^2 = t(\det g)^{-1/2}\cdot \det(C_g Z + D_g)$ for some $t \in \mathbf{C}$, $|t| = 1$. $\hat{G}_n$ is a multiplicative group under the multiplication $(g, \alpha(Z)) \cdot (h, \beta(Z)) = (gh, \alpha(h\langle Z\rangle)\beta(Z))$ and is called the universal covering group of $G_n$. This group was introduced by Shimura[S] for $n = 1$ and then generalized by Zhuravlev[Z1,2] for arbitrary $n$. Let $\gamma : \hat{G}_n \to G_n$ be the projection $\gamma(g, \alpha(z)) = g$. We define an action of $\hat{G}_n$ on $\mathcal{H}_n$ by

$$\zeta\langle Z\rangle = \gamma(\zeta)\langle Z\rangle$$

for $\zeta \in \hat{G}_n$, $Z \in \mathcal{H}_n$.

From now on, we let $q$ be a positive integer such that $4|q$. Let

$$(1.1) \qquad \theta^n(Z) = \sum_M e({}^t MMZ) = \sum_N e(Z[N]),\ Z \in \mathcal{H}_n,$$

where $M(N$, resp.) runs over all the integral row (column, resp.) matrices of length $n$. $\theta^n(Z)$ is called the standard theta-function of degree $n$. For $M \in \Gamma_0^n(q)$, we define

$$(1.2) \qquad j(M, Z) = \theta^n(M\langle Z\rangle)/\theta^n(Z), Z \in \mathcal{H}_n.$$

It is well known [Z1] that $(M, j(M, Z)) \in \hat{G}_n$. So the map $j : \Gamma_0^n(q) \to \hat{G}_n$ defined by $j(M) = (M, j(M, Z))$ is a well defined injective homomorphism such that $\gamma \circ j$ is the identity map on $\Gamma_0^n(q)$. We denote $\hat{\Gamma}_0^n(q) = j(\Gamma_0^n(q)), \hat{\Gamma}_0^n = j(\Gamma_0^n)$ and $\hat{L}_0^n = \gamma^{-1}(L_0^n)$.

Let $\chi$ be a Dirichlet character (mod $q$) and $k$ be a positive half integer, i.e., $k = m/2$ for some positive odd integer $m$. For a complex valued function $F$ on $\mathcal{H}_n$ and $\zeta = (g, \alpha(Z)) \in \hat{G}_n$, we set

$$(1.3) \qquad (F|_k\zeta)(Z) = r(g)^{(nk/2)-\langle n\rangle}\alpha(Z)^{-2k}F(g\langle Z\rangle),\ Z \in \mathcal{H}_n.$$

Since the map $Z \to g\langle Z\rangle$ is an analytic automorphism of $\mathcal{H}_n$ and $\alpha(Z) \neq 0$ on $\mathcal{H}_n$, $F|_k\zeta$ is holomorphic on $\mathcal{H}_n$ if $F$ is. Also from the definition follows that $F|_k\zeta_1|_k\zeta_2 = F|_k\zeta_1\zeta_2$ for $\zeta_1, \zeta_2 \in \hat{G}_n$.

A function $F : \mathcal{H}_n \to \mathbf{C}$ is called a Siegel modular form of degree $n$, weight $k$, level $q$, with character $\chi$ if the following conditions hold:

(i) $F$ is holomorphic on $\mathcal{H}_n$,

(ii) $F|_k \hat{M} = \chi(\det D_M) \cdot F$ for every $\hat{M} = (M, j(M,Z)) \in \hat{\Gamma}_0^n(q)$, and

(iii) $F|_k (M, \alpha(z))$ is bounded as $\operatorname{Im} z \to \infty$, $z \in \mathcal{H}_1$, for every $(M, \alpha(z)) \in \gamma^{-1}(\Gamma^1)$ when $n = 1$.

It is known[Kö] that the boundedness condition (iii) follows from (i) and (ii) for $n \geq 2$. We denote the set of all such Siegel modular forms by $\mathcal{M}_k^n(q, \chi)$. This is known[Si2] to be a finite dimensional vector space over $\mathbf{C}$.

A function $F : \mathcal{H}_n \to \mathbf{C}$ is called an even or odd modular form of degree $n$ if $F$ satisfies (i), (iii), and

(ii)′ $(\det D_M)^s F(M\langle Z \rangle) = F(Z)$, $Z \in \mathcal{H}_n$ for every $M \in \Gamma_0^n$, where $s = 0$ for even and $s = 1$ for odd modular forms.

We denote the set of all such even modular forms by $\mathcal{M}_0^n$ and odd modular forms by $\mathcal{M}_1^n$. They are also vector spaces over $\mathbf{C}$.

Let $F \in \mathcal{M}_k^n(q, \chi)$ and $\chi(-1) = (-1)^s$ for $s = 0$ or $1$. For $M \in \Gamma_0^n$, we have $\hat{M} = (M, j(M,Z)) = (M, 1)$ and $\det D_M = \pm 1$. So, $F$ satisfies (ii)′ and hence

$$\mathcal{M}_k^n(q, \chi) \subset \mathcal{M}_s^n \text{ if } \chi(-1) = (-1)^s. \tag{1.4}$$

Let $X = (\Gamma_0^n g \Gamma_0^n)$ with $g \in L_0^n$. Then $X$ can be written as a disjoint union of left cosets $(\Gamma_0^n g_i), g_i \in L_0^n, i = 1, 2, \ldots, \mu$. For $F \in \mathcal{M}_s^n$ and $X = (\Gamma_0^n g \Gamma_0^n)$, we set

$$F|_{k,\chi} X = \sum_{i=1}^{\mu} \chi(\det A_i) \cdot F|_k \tilde{g}_i, \tag{1.5}$$

where

$$\tilde{g}_i = (g_i, (\det g_i)^{-1/4} |\det D_i|^{1/2}) \in \hat{L}_0^n \tag{1.6}$$

with $g_i = \begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix} \in L_0^n$ and $\chi(-1) = (-1)^s$.

If we let $\mathcal{L}_0^n$ be the vector space over $\mathbf{C}$ formally spanned by double cosets $(\Gamma_0^n g \Gamma_0^n)$, $g \in L_0^n$, then we can extend the action (1.5) to $\mathcal{L}_0^n$ by linearity. It follows from definitions that for $F \in \mathcal{M}_s^n$ and $X, Y \in \mathcal{L}_0^n$, we have $F|_{k,\chi} X \in \mathcal{M}_s^n$ and $F|_{k,\chi} X|_{k,\chi} Y = F|_{k,\chi} XY$, where $\chi(-1) = (-1)^s$. So the action (1.5) is a well defined action of $\mathcal{L}_0^n$ on $\mathcal{M}_s^n$. The elements of $\mathcal{L}_0^n$ acting on $\mathcal{M}_s^n$ in this manner are called Hecke operators. It is well known[A2] that $\mathcal{L}_0^n$ is equipped with a ring structure and is called a Hecke ring.

## 2. Theta-Series and Theta Operators

Let $Q \in \mathcal{N}_m^+$. The level $q_Q$ of $Q$ is defined to be the smallest positive integer satisfying that $q_Q(2Q)^{-1}$ is integral with even diagonals. It is known that $q_Q$ is divisible by 4 if $m$ is odd. We define the theta-series of degree $n$ associated to $Q$ by

$$\theta^n(Z,Q) = \sum_{X \in M_{m,n}(\mathbf{Z})} e(Q[X]Z), \quad Z \in \mathcal{H}_n. \tag{2.1}$$

If we let $r(N,Q) = \left|\{X \in M_{m,n}(\mathbf{Z}) | Q[X] = N\}\right|$ for each $N \in \mathcal{N}_n$, then

$$\theta^n(Z,Q) = \sum_{N \in \mathcal{N}_n} r(N,Q) e(NZ), \quad Z \in \mathcal{H}_n. \tag{2.2}$$

It is known [K] that for odd $m$

$$\theta^n(Z,Q) \in \mathcal{M}_k^n(q_Q, \chi_Q) \tag{2.3}$$

where $k = m/2$ is a half integer and $\chi_Q$ is a Dirichlet character modulo $q_Q$ defined by

$$\chi_Q(M) = \left(\frac{2\det 2Q}{|\det D_M|}\right)_{Jac}, \quad {}^\forall M \in \Gamma_0^n(q). \tag{2.4}$$

For even $m$, (2.3) was proved by Andrianov and Maloletkin [A-M].

Let $\Theta_m^n$ be the vector space over $\mathbf{C}$ spanned by $\theta^n(Z,Q)$, $Q \in \mathcal{N}_m^+$ and let $\Theta_m^n(q,d)$ be its subspace spanned by $\theta^n(Z,Q)$, $Q \in \mathcal{N}_m^+$,

$\det 2Q = d$ and $q_Q = q$ for given positive integers $d, q$. It is easy to check that

$$\text{(2.5)} \qquad \Theta_m^n \subset \mathcal{M}_0^n \quad \text{and} \quad \Theta_m^n(q,d) \subset \mathcal{M}_{m/2}^n(q,\chi),$$

where $\chi(M) = \left(\dfrac{2d}{|\det D_M|}\right)_{Jac}$ for any $M \in \Gamma_0^n(q)$.

Let $\Theta_m^n[Q]$ be the subspace of $\Theta_m^n$ spanned by $\theta^n(Z, Q')$, $Q' \in gen(Q)$ where $gen(Q)$ is the genus of $Q$. Clearly

$$\text{(2.6)} \qquad \Theta_m^n[Q] \subset \Theta_m^n(q,d) \subset \Theta_m^n$$

if $q = q_Q$ and $d = \det 2Q$.

We now define theta operators following Andrianov[A2] : $p$ is still a prime relatively prime to $q$. Let $\alpha : L_0^m \to \mathbf{C}^\times$ be a character such that $\alpha(\Gamma_0^m) = 1$. For $X = (\Gamma_0^m g_0 \Gamma_0^m) \in \mathcal{L}_0^n$ with $g_0 = \begin{pmatrix} p^\delta D_0^* & B_0 \\ 0 & D_0 \end{pmatrix} \in L_0^m$ and $\theta^n(Z,Q) \in \Theta_m^n$, we set

$$\text{(2.7)} \qquad \theta^n(Z,Q) \circ_\alpha X = \alpha(g_0) \sum_{\substack{D \in \Lambda D_0 \Lambda / \Lambda \\ p^\delta Q[D^*] \in \mathcal{N}_m^+}} l_X(Q,D) \cdot \theta^n(Z, p^\delta Q[D^*])$$

where $\Lambda = SL_m(\mathbf{Z})$ and

$$\text{(2.8)} \qquad l_X(Q,D) = \sum_{B \in B_X(D)/\text{mod } D} e(QBD^{-1}).$$

Here

$$\text{(2.9)} \qquad B_X(D) = \left\{ B \in M_m(\mathbf{Q}) \; ; \; \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \in \Gamma_0^m g_0 \Gamma_0^m \right\}$$

and $B_1, B_2 \in B_X(D)$ are said to be congruent modulo $D$ on the right if $(B_1 - B_2)D^{-1} \in M_m(\mathbf{Z})$. This congruence is obviously an equivalent relation and the summation in (2.8) is over equivalent classes in $B_X(D)$

modulo $D$ on the right. We extend (2.7) by linearity to the whole space $\Theta_m^n$ and the whole ring $\mathcal{L}_0^m$.

Let

$$(2.10)\qquad \mathcal{L}_0^{m,0}=\mathcal{L}_{0,p}^{m,0}=\left\{\sum a_i(\Gamma_0^m g_i\Gamma_0^m)\in\mathcal{L}_0^m\ ;\ \delta_i m-2b_i=0\right\}$$

where $g_i=\begin{pmatrix}p^{\delta_i}D_i^* & B_i\\ 0 & D_i\end{pmatrix}\in L_0^m$ and $b_i=\log_p|\det D_i|$, and let

$$(2.11)\qquad \mathcal{E}_0^{m,0}=\mathcal{E}_{0,p}^{m,0}=\left\{\sum a_i(\Gamma_0^m g_i\Gamma_0^m)\in\mathcal{L}_0^{m,0}\ ;\ \delta_i\in 2\mathbf{Z}\right\}.$$

## 3. Main Theorem

We prove the following theorem:

THEOREM 3.1. (1) *The action (2.7) is a well-defined action of $\mathcal{L}_0^m$ on $\Theta_m^n$. The elements of $\mathcal{L}_0^m$ acting on $\Theta_m^n$ in this manner are called theta operators.*

(2) *$\Theta_m^n(q,d)$ is invariant under the theta operators of $\mathcal{L}_0^{m,0}$ if $p$ and $q$ are relatively prime..*

(3) *$\Theta_m^n[Q]$ is invariant under the theta operators of $\mathcal{E}_0^{m,0}$ if $p$ and $2q_Q$ are relatively prime.*

*Proof.* This theorem is proved for the case $m$ even in [A1]. So, we restrict ourselves to the case $m$ odd here. Let

$$(3.1)\qquad \varepsilon(Z,Q)=\sum_{U\in\Omega}e(Q[U]Z),\quad Z\in\mathcal{H}_m,$$

where $\Omega=GL_m(\mathbf{Z})$.

$\varepsilon(Z,Q)$ is called the $\varepsilon$-series of $Q$. For every $M=\begin{pmatrix}D^* & B\\ 0 & D\end{pmatrix}\in\Gamma_0^m$ with $D\in\Omega$, we have

$$(3.2)\qquad \varepsilon(M\langle Z\rangle,Q)=\sum_{U\in\Omega}e(Q[UD^*]Z)\cdot e(Q[U]BD^{-1})=\varepsilon(Z,Q)$$

Note that $e(Q[U]BD^{-1}) = 1$ because $Q[U] \in \mathcal{N}_m^+$ and $BD^{-1}$ is integral symmetric [M]. From (3.2) and the definition of even modular forms follows that

$$\varepsilon(Z,Q) \in \mathcal{M}_0^m.$$

Let

$$\mathcal{A}_m = \left\{\sum c_i \varepsilon(Z,Q_i)\ ;\ Q_i \in \mathcal{N}_m^+\right\} \subset \mathcal{M}_0^m. \tag{3.3}$$

Let $k = m/2$ and $\chi$ be a character satisfying $\chi(-1) = 1$. Let $X = (\Gamma_0^m g_0 \Gamma_0^m) \in \mathcal{L}_0^m$ with $g_0 = \begin{pmatrix} p^\delta D_0^* & B_0 \\ 0 & D_0 \end{pmatrix} \in L_0^m$. Then

$$X = \sum_{\substack{D \in \Omega\backslash\Omega D_0 \Omega \\ B \in B_\chi(D)/\mathrm{mod}\, D}} (\Gamma_0^m g),$$

where $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \in L_0^m$. See (2.9) for $B_\chi(D)/\mathrm{mod}\, D$. From (1.3) and (1.6) follows

$$\varepsilon(Z,Q) = \sum_{U \in \Omega} e(Q[U]Z) = \sum_{U \in \Omega} e(QZ)|_k \tilde{M}_U \tag{3.4}$$

where $M_U = \begin{pmatrix} U^* & 0 \\ 0 & U \end{pmatrix} \in \Gamma_0^m$ and $\tilde{M}_U = (M_U, 1)$. Hence

$$\varepsilon(Z,Q)|_{k,\chi} X = \sum_{U \in \Omega} \sum_{\substack{D \in \Omega\backslash\Omega D_0 \Omega \\ B \in B_\chi(D)/\mathrm{mod}\, D}} \chi(\det p^\delta D^*) \cdot e(QZ)|_k \tilde{M}_U |_k \tilde{g}, \tag{3.5}$$

where $\tilde{g} = (g, p^{-\delta m/4} |\det D|^{1/2})$. Since $M_U g = \begin{pmatrix} p^\delta (UD)^* & U^* B \\ 0 & UD \end{pmatrix}$ and $U^*\{B_\chi(D)/\mathrm{mod}\, D\} = \{B_\chi(UD)/\mathrm{mod}\, UD\}$ for any $U \in \Omega$, we have

$$\varepsilon(Z,Q)|_{k,\chi} X = \sum_{\substack{D \in \Omega D_0 \Omega \\ B \in B_\chi(D)/\mathrm{mod}\, D}} \chi(\det p^\delta D^*) \cdot e(QZ)\,|_k\, \tilde{g}$$

So, we may rewrite (3.5) as

$$(3.6)\quad \varepsilon(Z,Q)|_{k,\chi}X = \sum_{\substack{D\in\Omega D_0\Omega/\Omega \\ B\in B_\chi(D)/\mathrm{mod}\, D}} \sum_{U\in\Omega} \chi(\det p^\delta D^*)\cdot e(QZ)|_k\tilde{g}|_k\tilde{M}_U .$$

We now consider

$$(3.7)\qquad \beta(Z,Q) = \sum_{B\in B_\chi(D)/\mathrm{mod}\, D} \chi(\det p^\delta D^*)\cdot e(QZ)|_k\tilde{g}.$$

From (1.3) follows that

$$\begin{aligned}
&\beta(Z,Q)\\
&= \sum_{B\in B_\chi(D)/\mathrm{mod}\, D} \chi(p^{\delta m-b})p^{\delta(mk-\langle m\rangle)-bk} e(Qp^\delta Z[D^{-1}] + QBD^{-1})\\
&= \chi(p^{\delta m-b})p^{\delta(mk-\langle m\rangle)-bk} e(p^\delta Q[D^*]Z) \sum_{B\in B_\chi(D)/\mathrm{mod}\, D} e(QBD^{-1}),
\end{aligned}$$

and so,

$$(3.8)\quad \beta(Z,Q) = \alpha_{k,\chi}(g_0)\cdot e(p^\delta Q[D^*]Z) \sum_{B\in B_\chi(D)/\mathrm{mod}\, D} e(QBD^{-1}),$$

where $b = \log_p|\det D| = \log_p|\det D_0|$ and $\alpha_{k,\chi} : L_0^m \to \mathbf{C}^\times$ is a character defined by

$$(3.9)\qquad \alpha_{k,\chi}(g) = \chi(p^{\delta m-b})p^{\delta(mk-\langle m\rangle)-bk}$$

for any $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D\end{pmatrix} \in L_0^m$. If we take $B + AD$ instead of $B$ as a representative in $B_\chi(D)/\mathrm{mod}\, D$ where ${}^tA = A \in M_m(\mathbf{Z})$, then $e(Q(B+AD)D^{-1}) = e(QBD^{-1})\cdot e(QA) = e(QBD^{-1})$. So (3.8) is independent of the choice of representatives $B$ of $B_\chi(D)/\mathrm{mod}\, D$. Let $g_S = \begin{pmatrix} I_m & S \\ 0 & I_m\end{pmatrix} \in \Gamma_0^m$ with ${}^tS = S \in M_m(\mathbf{Z})$. Then $\tilde{g}_S = (g_S, 1)$

and $gg_s = \begin{pmatrix} p^\delta D^* & p^\delta D^* S + B \\ 0 & D \end{pmatrix}$ so that $\{B + p^\delta D^* S\}$ is a complete set of representatives of $B_\chi(D)/\text{mod}D$ if $\{B\}$ is. Therefore, from (3.7) follows

$$\beta(Z,Q)|_k \tilde{g}_s = \beta(Z,Q)$$

Applying $|_k \tilde{g}_s$ on the right hand side of (3.8), we obtain

$$\beta(Z,Q) = \alpha_{k,\chi}(g_0) \cdot e(p^\delta Q[D^*]Z) \cdot e(p^\delta Q[D^*]S) \cdot l_\chi(Q,D).$$

So, if $l_\chi(Q,D) \neq 0$, then $e(p^\delta Q[D^*]S) = 1$ for any ${}^tS = S \in M_m(\mathbf{Z})$. This implies that $p^\delta Q[D^*] \in \mathcal{N}_m^+$. In other words, if $p^\delta Q[D^*] \notin \mathcal{N}_m^+$, then $l_\chi(Q,D) = 0$. From this and (3.4), (3.6), (3.8) follows

$$\varepsilon(Z,Q)|_{k,\chi} X = \alpha_{k,\chi}(g_0) \sum_{\substack{D \in \Omega D_0 \Omega/\Omega \\ p^\delta Q[D^*] \in \mathcal{N}_m^+}} \varepsilon(Z, p^\delta Q[D^*]) \cdot l_\chi(Q,D) \in \mathcal{A}_m.$$

Choosing a complete set of representatives $\{D_i\}$ of $\Omega D_0 \Omega/\Omega$ such that $\det D_i = \det D_0$, we may rewrite the above as follows :

$$\text{(3.10)} \quad \varepsilon(Z,Q)|_{k,\chi} X = \alpha_{k,\chi}(g_0) \sum_{\substack{D \in \Lambda D_0 \Lambda/\Lambda \\ p^\delta Q[D^*] \in \mathcal{N}_m^+}} \varepsilon(Z, p^\delta Q[D^*]) \cdot l_\chi(Q,D).$$

We now define a linear map $\vartheta = \vartheta_{m,n} : \mathcal{A}_m \to \Theta_m^n$ by $\vartheta(\varepsilon(Z,Q)) = \theta^n(Z_n,Q)$, $Q \in \mathcal{N}_m^+$, where $Z = \begin{pmatrix} Z_n & * \\ * & * \end{pmatrix} \in \mathcal{H}_m, Z_n \in \mathcal{H}_n$. Obviously $\vartheta$ is a well-defined epimorphism. From (3.10) and (2.7) follows

$$\vartheta(\varepsilon(Z,Q)\,|_{k,\chi} X) = \theta^n(Z,Q) \circ_\alpha X,$$

for any $X \in \mathcal{L}_0^m$, where $\alpha = \alpha_{k,\chi}$ as in (3.9).

Now $\varepsilon(Z,Q)|_{k,\chi} X\,|_{k,\chi} Y = \varepsilon(Z,Q)|_{k,\chi} XY$ implies

$$\text{(3.11)} \qquad \theta^n(Z,Q) \circ_\alpha X \circ_\alpha Y = \theta^n(Z,Q) \circ_\alpha XY.$$

From the surjectivity of $\vartheta$, (3.10) and (3.11) follows (1).

Let $p$ be relatively prime to $q$ and let $X \in \mathcal{L}_0^{m,0}$. We may assume $X = (\Gamma_0^m g \Gamma_0^m)$, $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \in L_0^m$ such that $\delta m = 2b$ with $b = \log_p |\det D|$.

To prove (2), it is enough to show that $\det 2Q' = d$ and $q_{Q'} = q$ for $Q' = p^\delta Q[D^*] \in \mathcal{N}_m^+$, where $d = \det 2Q$ and $q = q_Q$. Clearly $\det 2Q' = d$. Let $q'$ be the level of $Q'$. Then $q(2Q')^{-1}p^{\delta'} = qp^{\delta'-\delta}(2Q)^{-1}[D]$ is integral for some $\delta' \geq 0$. So. $q' \mid qp^{\delta'}$, which implies $q' \mid q$. Similary $q \mid q'$. This proves (2).

Finally, let $\delta$ be even. Note that since we restrict ourselves to the case $m$ odd, $q = q_Q$ is divisible by 4. Let $D' = p^{\delta/2}D^*$ so that $\det D' = \pm 1$. It is a well known fact[Og] that $q$ and $d = \det(2Q)$ have the same prime factors. So $p \mid d$ and hence one can find $U \in M_m(\mathbf{Z})$ such that $U \equiv D' (\bmod 8d^3)$. Since $2Q' = 2Q[D']$, we have $2Q' \equiv 2Q[U] \pmod{8d^3}$. Therefore[Si2] $Q' \in gen(Q)$ if $Q' = p^\delta Q[D^*] \in \mathcal{N}_m^+$ and this proves (3).

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