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A NOTE ON TOTALLY GEODESIC MAPS

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1. Introduction

Let $f : M \to N$ be a smooth map between Riemannian manifolds M and N. If f maps geodesics of M to geodesics of N, f is called *totally geodesic*. As is well known, totally geodesic maps are harmonic and the image $f(M)$ of a totally geodesic map $f : M \to N$ is an immersed totally geodesic submanifold of N (cf. § 6.3 of [W]). We are interested in the following question: When is a harmonic map $f : M \to N$ with rank ≤ 1 everywhere on M totally geodesic? In other words, when is the image of a harmonic map $f : M \to N$ with rank ≤ 1 everywhere on M geodesics of N? In this note, we give some sufficient conditions on curvatures of M. It is interesting that no curvature assumptions on target manifolds are necessary in Theorems 1 and 2. Some properties of totally geodesic maps are also given in Theorem 3. We think our Theorem 3 is somewhat unusual in view of the following classical theorem of Eells and Sampson (see p.124 of [ES]).

THEOREM (EELLS AND SAMPSON). *Let M be compact with nonnegative Ricci curvature and the sectional curvature of N is nonpositive. Then every harmonic map $f : M \to N$ is totally geodesic.*

2. Results

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We use the following Weitzenböck formula for harmonic maps (cf. [EL]) $f : M \to N$;

(1)
$$\frac{1}{2}\triangle\|df\|^2 = -\|\nabla df\|^2 + \sum_{i,j}\langle R^N(df(e_i), df(e_j))df(e_i), df(e_j)\rangle - \sum_i \langle df(Ric^M(e_i)), df(e_i)\rangle$$

where $\{e_i\}$ is an orthonormal basis of the tangent space of M at the point considered, ∇df is the second fundamental form of f. Note that our Laplacian $\triangle$ is the negative of the usual Laplacian acting on functions. Recall that f is totally geodesic if $\nabla df \equiv 0$ and f is harmonic if $(\text{trace}\nabla df) \equiv 0$.

THEOREM 1. *Let M be compact with nonnegative Ricci curvature. Then every harmonic map $f : M \to N$ with rank of $f \leq 1$ everywhere on M is totally geodesic.*

Proof. Because of the rank condition, the second term of the right-hand side of (1) is identically zero and by the curvature assumption, the inequality

$$(2) \qquad \sum_i \langle df(Ric^M(e_i)), df(e_i)\rangle \geq -a^2 \sum_i \|df(e_i)\|^2 = -a^2\|df\|^2$$

holds for every real number a. Then we have

$$(3) \qquad \frac{1}{2}\triangle\|df\|^2 \leq -\|\nabla df\|^2 + a^2\|df\|^2.$$

By the Divergence Theorem, we have

$$(4) \qquad 0 \leq \int_M \|\nabla df\|^2\, dM \leq a^2 \int_M \|df\|^2\, dM.$$

Now, taking $a \to 0$ we get $\|\nabla df\|^2 \equiv 0$, that is, f is totally geodesic.

COROLLARY 1.1. *Let M be compact with the Ricci curvature bounded below by $a^2 > 0$. Then every harmonic map with rank of $f \leq 1$ everywhere on M is constant.*

Proof. Proceeding as in the proof of Theorem 1, in this case we get

$$(5) \qquad 0 \leq \int_M \|\nabla df\|^2 \, dM \leq -a^2 \int_M \|df\|^2 \, dM.$$

Since $a \neq 0$, it must be that $\|df\| \equiv 0$, that is, f is constant.

For noncompact complete Riemannian manifolds, we have the following

THEOREM 2. *Let M be complete noncompact with nonnegative Ricci curvature. Then any harmonic map $f : M \to N$ with rank of $f \leq 1$ everywhere on M is constant if its energy integral is finite.*

Proof. Set $\epsilon(f) = \|df\|^2$. Then by (1), our assumptions on the curvature and the rank of f imply

$$(6) \qquad \triangle \epsilon(f) \leq -2 \|\nabla df\|^2.$$

Now, the proof of Theorem 2, section 6.4 of [W] completes our proof.

Without the rank condition, we have the following result for totally geodesic maps;

THEOREM 3. *Let M be a compact Riemannian manifold with the Ricci curvature bounded above by $-a^2 < 0$ and let N be a Riemannian manifold with nonnegative sectional curvature. Then every totally geodesic map $f : M \to N$ is constant.*

Proof. Since f is totally geodesic, our curvature assumptions and (1) imply

$$\frac{1}{2} \triangle \|df\|^2 \geq a^2 \|df\|^2.$$

By the Divergence Theorem, we have

$$0 \geq a^2 \int_M \|df\|^2 \, dM.$$

Since $a \neq 0$, $\|df\|^2 \equiv 0$, i.e., f is constant.

References

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