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ON THE ASYMPTOTIC–NORMING PROPERTY IN LEBESGUE–BOCHNER FUNCTION SPACES

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1. Introduction

In [12] Zhibao Hu and Bor-Luh Lin proved that a Banach space X has the asymptotic-norming property I if and only if $L_p(X,\mu)$, $1<p<\infty$, has the asymptotic-norming property I. And they proved that if X is an Asplund space and X has the asymptotic-norming property III, then $L_p(X,\mu)$, $1<p<\infty$ also has the asymptotic-norming property III.

In this paper we prove that if (Ω,Σ,μ) is a non-purely atomic measure space and X is strictly convex, then X has the asymptotic-norming property II if and only if $L_p(X,\mu)$, $1<p<\infty$, has the asymptotic-norming property II. And we prove that if X^* is an Asplund space and strictly convex, then for any p, $1<p<\infty$,

X^* has the w^*-ANP-II if and only if $L_p(X^*,\mu)$ has the w^*-ANP-II.

2. Preliminaries and definitions

Throughout this paper, let X be a Banach space, $S_X=\{x : x\in X, \|x\|=1\}$ and $B_X=\{x : x\in X, \|x\|\le 1\}$.

DEFINITION 2.1([8,11]). Let Φ be a subset of B_{X^*}. Φ is a norming set of X if $\|x\|=\sup_{x^*\in\Phi} x^*(x)$ for all x in X. A sequence $\{x_n\}$ in S_X is said to be asymptotically normed by Φ if for any $\epsilon>0$, there is $x^*\in\Phi$ and $m\in N$ such that $x^*(x_n)>1-\epsilon$ for all $n\ge m$.

DEFINITION 2.2([11,12]). For $\kappa=$ I, II, or III, a sequence $\{x_n\}$ in X is said to have the property k if

(I) $\{x_n\}$ is convergent;
(II) $\{x_n\}$ has a convergent subsequence; or
(III) $\cap_{n=1}^{\infty}\overline{\text{co}}\{x_k : k\ge n\}\neq\emptyset$.

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Let Φ be a norming set of X. X is said to have the asymptotic-norming property $\kappa, \kappa =$ I, II, III, with respect to Φ(Φ-ANP-κ) if every sequence in S_X that is asymptotically normed by Φ has the property κ. X is said to have the asymptotic-norming property κ(ANP-κ), if there is an equivalent norm $\|\cdot\|$ on X such that there is a norming set Φ with respect to $(X, \|\cdot\|)$ such that X has the Φ-ANP-κ, where $\kappa =$ I, II, III.

The following Theorem is proved in [11].

THEOREM 2.3. *Let Φ be a norming set of X. Then*

(a) *X has the Φ-ANP-I if and only if X has the Φ-ANP-III and X has the property (G), i.e. every point of S_X is a denting point of B_X [3,4];*
(b) *X has the Φ-ANP-II if and only if X has the Φ-ANP-III and X has the Kadec-Klee property, i.e. for any sequence $\{x_n\}$ and $\{x\}$ in B_X such that $\lim_n \|x_n\| = \|x\| = 1$ and $w-\lim_n x_n = x$ then $\lim_n \|x_n - x\| = 0$ [1,2];*
(c) *X has the Φ-ANP-I if and only if X has the Φ-ANP-II and X is strictly convex, i.e. S_X contains no non-trivial line segments [5].*

The following Theorem is proved in [12].

THEOREM 2.4. *Let X be a separable Banach space and let (Ω, Σ, μ) be a σ-finite measure space. Let Φ be a norming set of X such that $\Phi = co\{\Phi \cup \{0\}\}$ and $\Phi \cap S_{X^*} = \emptyset$. Let $\triangle(\Phi, \Sigma, \mu) = \bigcup_{n=1}^{\infty} \triangle_n$ where*

$$\triangle_n = \left\{ \sum_{i=1}^{m} \lambda_i x_i^* \chi_A : \lambda_i \geq 0,\ x_i \in \Phi,\ \frac{n-1}{n} \leq \|x_i^*\| < \frac{n}{n+1},\right.$$
$$\left. A_i \cap A_j = \emptyset \text{ for all } i \neq j,\ i,j = 1, \cdots, m \text{ and } \sum_{i=1}^{m} \lambda_i^q \mu(A_i) = 1 \right\}.$$

Then

(a) *$\triangle(\Phi, \Sigma, \mu)$ is a norming set of $L_p(X, \mu)$;*
(b) *If X is an Asplund space [9] and X has the Φ-ANP-III, then $L_p(X, \mu)$ has the $\triangle(\Phi, \Sigma, \mu)$-ANP-III.*

The following Theorem is proved in [12].

THEOREM 2.5. *Let X be a Banach space and let (Ω, Σ, μ) be a measure space. Then for any $p, 1 < p < \infty$,*
X has the ANP-I if and only if $L_p(X, \mu)$ has the ANP-I.

3. Asymptotic-norming property in $L_p(X, \mu)$ $(1 < p < \infty)$

In this section we prove that if (Ω, Σ, μ) is a non-purely atomic measure space and X is strictly convex, then X has the ANP-II if and only if $L_p(X, \mu)$ has the ANP-II. And we prove that if X^* is strictly convex and Asplund, then X^* has the w^*-ANP-II if and only if $L_p(X^*, \mu)$ has the w^*-ANP-II.

LEMMA 3.1. *Let (Ω, Σ, μ) be a non-purely atomic measure space. Then the following are equivalent:*

(i) *$L_p(X, \mu)$ has the ANP-I;*
(ii) *$L_p(X, \mu)$ has the ANP-II.*

Proof. Suppose $L_p(X, \mu)$ has the Φ-ANP-II for some norming set Φ in the dual space of $L_p(X, \mu)$. Then $L_p(X, \mu)$ has the Kadec-Klee property by Theorem 2.3. Since (Ω, Σ, μ) is non-purely atomic, it follows from [10] that $L_p(X, \mu)$ is strictly convex. By Theorem 2.3, $L_p(X, \mu)$ has the ANP-I.

COROLLARY 3.2. *Let (Ω, Σ, μ) be a non-purely atomic measure space and let X be a Banach space having the property (G), then the following are equivalent:*

(i) *$L_p(X, \mu)$ has the ANP-I;*
(ii) *$L_p(X, \mu)$ has the ANP-II;*
(iii) *$L_p(X, \mu)$ has the ANP-III.*

Proof. We only need to show that (iii) implies (i). Let $L_p(X, \mu)$ has the Φ-ANP-III for some norming set Φ in the dual space of $L_p(X, \mu)$. Since X has (G), $L_p(X, \mu)$ has (G)[1] and thus $L_p(X, \mu)$ has the Φ-ANP-I by Theorem 2.3. Hence $L_p(X, \mu)$ has the ANP-I.

THEOREM 3.3. *Let X be a Banach space and let (Ω, Σ, μ) be a non-purely atomic measure space. Let X be strictly convex. Then for any p, $1 < p < \infty$,*

X has the ANP-II if and only if $L_p(X,\mu)$ has the ANP-II.

Proof. Let $L_p(X,\mu)$ has the ANP-II. Then by Lemma 3.1 $L_p(X,\mu)$ has the ANP-I. By Theorem 2.5 X has the ANP-I and so X has the ANP-II.

Conversely, suppose X has the Φ-ANP-II for some norming set Φ of B_{X^*}. Since X is strictly convex, X has the Φ-ANP-I by Theorem 2.3. By Theorem 2.5 $L_p(X,\mu)$ has the ANP-I. Hence $L_p(X,\mu)$ has the ANP-II.

DEFINITION 3.4[11]. Let X^* be a dual Banach space of X. We say that X^* has the weak* asymptotic-norming property κ(w^*-ANP-κ) if there is an equivalent norm $\|\cdot\|$ on X and a norming set Φ in $B_{(X,\|\cdot\|)}$ such that $(X^*,\|\cdot\|)$ has the Φ-ANP-κ, $\kappa=$ I, II, or III.

The following Corollary is in [12].

COROLLARY 3.5. *Let (Ω,Σ,μ) be a measure space. If X^* has the w^*-ANP-κ, $\kappa=$ I or III and X^* is an Asplund space then $L_p(X^*,\mu)$, $1<p<\infty$, has the w^*-ANP-κ, $\kappa=$ I or III.*

LEMMA 3.6. *Let X be a Banach space which X^* is Asplund and let (Ω,Σ,μ) be a measure space. If X^* has the Φ-ANP-III for some norming set Φ of B_X, then $L_p(X^*,\mu)$ has the $\triangle(\Phi,\Sigma,\mu)$-ANP-III, where $\triangle(\Phi,\Sigma,\mu)$ be defined as in Theorem 2.4.*

Proof. If X^* has the Φ-ANP-III for some norming set Φ in B_X, then we may assume that $\Phi=\text{co}\{\Phi\cup\{0\}\}$ and $\Phi\cap S_X=\emptyset$. Suppose $\{f_n\}$ be a sequence in the unit sphere of $L_p(X^*,\mu)$ that is asymptotically normed by $\triangle(\Phi,\Sigma,\mu)$. Since each f_n is essentially separably valued, we may assume that there is a separable subspace of Y of X^* such that $f_n(\Omega)\subset Y$ for all $n\in N$. Let $\Omega_1=\{t\in\Omega \mid f_n(t)\neq 0 \text{ for some } n\in N\}$, $\sum_1=\{A\cap\Omega_1 \mid A\in\sum\}$ and $\mu_1=\mu\,|_{\sum_1}$. Then $(\Omega_1,\Sigma_1,\mu_1)$ is a σ-finite measure space. Let $\Phi_1=\{x\,|_Y\colon x\in\Phi\}$. Then $\Phi_1=\text{co}\{\Phi_1\cup\{0\}\}$ and $\Phi_1\cap S_Y=\emptyset$ and Y has the Φ_1-ANP-III and Y^* is separable and so Y^* has the RNP[6], i.e. Y is an Asplund space. By Theorem 2.4, it follows that $L_p(Y,\mu_1)$ has the $\triangle(\Phi_1,\Sigma_1,\mu_1)$-ANP-III. Hence $L_p(X^*,\mu)$ has the $\triangle(\Phi,\Sigma,\mu)$-ANP-III.

THEOREM 3.7. *Let X^* be an Asplund space and let X^* be strictly convex. Then for any p, $1 < p < \infty$,*

X^* has the w^*-ANP-II if and only if $L_p(X^*, \mu)$ has the w^*-ANP-II.

Proof. We only need to show that if X^* has the w^*-ANP-II then $L_p(X^*, \mu)$ has the w^*-ANP-II. Suppose X^* has the w^*-ANP-II. Then X^* has the RNP. Hence $L_p(X^*, \mu) = [L_q(X, \mu)]^*$[6]. If X^* has the Φ-ANP-II for some norming set Φ in B_X, then X^* has the Φ-ANP-III and X^* has the Kadec-Klee property. Since X^* has the Φ-ANP-III and X^* is Asplund, by Lemma 3.6 $L_p(X^*, \mu)$ has the $\triangle(\Phi, \Sigma, \mu)$-ANP-III. Also $\triangle(\Phi, \Sigma, \mu)$ is in the unit ball of $L_q(X, \mu)$. Since X^* is a strictly convex space with the Kadec-Klee property, $L_p(X^*, \mu)$ has the Kadec-Klee property [1]. Hence, by Theorem 2.3, $L_p(X^*, \mu)$ has the $\triangle(\Phi, \Sigma, \mu)$-ANP-II and so $L_p(X^*, \mu)$ has the w^*-ANP-II.

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