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CARLEMAN INEQUALITIES WITH SAME EXPONENT

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1. Introduction

It has been shown [2] that the following Carleman inequality

(1) $\quad \|e^{t\phi}\nabla f\|_{L^q(U\setminus\{0\},dx)} \le C\|e^{t\phi}\Delta f\|_{L^p(U\setminus\{0\},dx)} \quad f \in C_0^\infty(U\setminus\{0\})$

holds for $(p,q) = \left(\frac{6n-4}{3n+2}, 2\right)$ and also for the pair $\left(2, \frac{6n-4}{3n-6}\right)$. His idea has been adopted by main stream of mathematicians to prove unique continuation theorem. In [2] the author showed the following:

When u and v are solutions of the differential equation

(2) $$\left(\Delta + \sum a_j \partial/\partial x_j + b\right) u = 0$$

in an non-empty connected open subset of R^n, where $a_j \in L^r_{\text{loc}}(R^n)$, $b \in L^s_{\text{loc}}(R^n)$ for $r = \frac{3n-2}{2}$ and $S > \frac{n}{2}$.

Then if $u - v$ vanishes to infinite order at a point, then $u = v$ identically. As we can see from [2] the exponents satisfy $\frac{1}{p} - \frac{1}{q} = \frac{1}{r} = \frac{2}{3n-2}$.

But in this type of Sobolev inbedding theorem $\frac{1}{n}$ is the natural exponent we can expect. So there has been a gap between two numbers. In the course of study the author became interested in the maximum range of (p,q)'s such that (1) holds. In general this inequality (1) does not hold for $p = q$ unless $p = q = 2$ since the corresponding restriction

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theorem is not true. In this paper we try to find the range of p's such that the inequality is true. After that using the interpolation theorem we can fine the range of (p,q) such that (1) is true.

2. Statements of results

THEOREM 2.1. *Let $n \geq 3$, $\frac{6n-4}{3n+2} \leq p \leq \frac{6n-4}{3n-6}$. There is a constant C depending only on n such that for all $t \in R$*

(3)

$$\|e^{t\phi} f\|_{L^p(U\setminus\{0\})} \leq C\|e^{t\phi} Df\|_{L^p(U\setminus\{0\})} \text{ for all } f \in C_0^\infty(U\setminus\{0\} : C^m).$$

3. Dirac operator

The Dirac operator is a first-order constant coefficient operator on R^n of the form $D = \sum \alpha_j \partial/\partial x_j$, where $\alpha_1, \cdots, \alpha_n$ are skew hermitian matrices satisfying the Clifford relations:

$$\alpha_j^* = -\alpha_j, \text{ and } \alpha_j\alpha_k + \alpha_k\alpha_j = -2\delta_{jk};\ j,k = 1, \ldots n.$$

Also $D^2 = -\Delta$

At this moment we want to introduce polar coordinates: $x = e^y w$, $y = \log|x|$, $w = x/|x| \in S$. The operator $L = \sum_{j<k} \alpha_j\alpha_k(x_j\partial/\partial x_k - x_k\partial/\partial x_j)$ acts only in the w-variable. Then after simple computation

$$e^y D = \hat{\alpha}(\partial/\partial y - L) \text{ for } \hat{\alpha} = \sum_j \alpha_j x_j/|x|.$$

Also L satisfies

$$L(L+n-2) = -\Delta_S \tag{4}$$

where Δ_S denotes the Laplace-Beltrami operator of the sphere. Also the following relation holds [1].

$$e^{t\psi(y)} e^y D e^{-t\psi(y)} h = \hat{\alpha} A_t h \tag{5}$$

where

$$A_t = \partial/\partial y - (t\psi'(y) + L).$$

Throughout this paper $\phi(x) = \psi(y)$, $y = -\psi(y) + e^{-\varepsilon\psi(y)}$, for some $\varepsilon < 1/2$, $y < 0$ (since we are interested in the region $|x| < 1$) given implicetly as in [2]. The reason for using this particular weight function is because of convexity.

4. Proof

From the definition of $\psi(y)$, we get

$$\psi'(y) = -1/1 + \varepsilon e^{-\varepsilon\psi(y)} < 0.$$

We also find

$$\psi''(y) = \varepsilon^2 e^{-\varepsilon\psi(y)}/(1 + \varepsilon e^{-\varepsilon\psi(y)})^3 \geq c e^{\varepsilon y}.$$

First we will try to find the range of p such that the following inequality is true.

$$\|f\|_{L^q(U\setminus\{0\})} \leq C\|A_t f\|_{L^p(U\setminus\{0\})} \text{ for } f \in C_0^\infty(U\setminus\{0\} : C^m).$$

We can rewrite

$$A_t f = \sum_k (\partial/\partial y - (t\psi'(y) + k))\pi_k f = \sum A_{t,k} f.$$

To prove the above we found a left inverse operator B_t of A_t satisfying $f(y) = B_t A_t f(y)$ [2] such that

$$B_{1,t} f(y, w) = \sum \int \sigma_{1,t}(y, \eta, k)\hat{f}(\eta, .)(w) e^{iy\eta} d\eta \pi_k$$

where

$$\sigma_{1,t}(y, \eta, k) = \int_{-\infty}^{y} e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds$$

which converges only for $k < -t\psi'(y)$.
and

$$B_{2,t} f(y, w) = \sum \int \sigma_{2,t}(y, \eta, k)\hat{f}(\eta, .)(w) e^{iy\eta} d\eta \pi_k$$

where

$$\sigma_{2,t}(y, \eta, k) = -\int_{y}^{\infty} e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds$$

which converges only for $k > -t\psi'(y)$.

We can extend these symbols by Stein's method [7, p.128] to the whole range and then the extended symbols satisfy the following estimate [2].

$$(6)\quad \left\|(\frac{\partial}{\partial y})^N(\frac{\partial}{\partial \eta})^M d_m^k \sigma_{i,t}(y,\eta,k)\right\| \le C\frac{C_{M,N,m}}{(\sqrt{a}+|t\psi'(y)+k--i\eta|)^{M+N+1}} \times \left(1+\frac{t\psi'(y)}{\sqrt{a}+|t\psi'(y)+k-i\eta|}\right)^N .$$

Now the idea is to prove slightly stronger estimate i.e., L^p to L^p estimate. In this context we want to prove the following.

$$\|B_{i,t}f\|_{L^p(L^2(S,dw),dy)} \le C\|f\|_{L^p(R^-\times S,dydw)} \quad i=1,2.$$

The main tool in this proof is the spherical restriction theorem of C.Sogge [5].

LEMMA 4.1. *Let ξ_k denote the projection operator from $L^2(S)$ to the space of spherical harmonics of degree k. Then there is a constant c shch that*

$$\|\xi_k g\|_{L^{p'}(S)} \le ck^{1-2/n}\|g\|_{L^p(S)} \quad \text{where } p=\frac{2n}{2n+2},\ p'=\frac{2n}{2n-2}.$$

Then for $T=\operatorname{sgn}\left(L+\frac{n-2}{2}\right)$, which is a bounded operator from $L^q(S;C^m)$ to $L^q(S;C^m)$ for all $q, 1<q<\infty$, let

$$\pi_k = \frac{1}{2}(1+T)\xi_k,\quad k=0,1,2,\dots$$
$$\pi_k = \frac{1}{2}(1-T)\xi_k,\quad k=1-n,-n,-n-1,\dots$$

Define $\pi_{M,N}$ by

$$\pi_k\pi_{M,N}g = \{\pi_k g \text{ if } M\le k\le N,\ 0 \text{ otherwise}\}.$$

Then using a device due to P.Tomas it is known [1]

(7) $$\|\pi_{M,N}g\|_{L^q(S;C^M)} \leq C(N^{\frac{n-2}{2}}(N-M+1)^{n/2})^{\frac{1}{2}-\frac{1}{q}}\|g\|_{L^2(S;C^m)}$$

for $2 \leq q \leq p' = 2n/2n-2$.(In particular this holds for $q = \dfrac{6n-4}{3n-6}$).

Our main ideas are as follows. We will try to obtain an local estimate for $y \in (-j, -j+1)$, for some integer j. After that we can extend the result to the whole range using partition of unity. Here convexity of $\psi(y)$ plays role. Let N be the integer satisfying

$$2^{N-1} \leq 10e^{\varepsilon j}t^{1/2} \leq 2^N.$$

Consider a partition of unity $\{\phi_\beta\}_0^N$ of the positive real axis satisfying

$$\sum_0^N \phi_\beta(r) = 1 \text{ all } r > 0$$

$$\operatorname{supp}\phi_\beta \sim \{r : 2^{\beta-2} \leq r \leq 2^\beta\}\ \beta = 1, ..., N-1$$

$$\operatorname{supp}\phi_0 \sim \{r : r \leq 1\},\ \operatorname{supp}\phi_N \sim \{r : r \geq \sqrt{a}/400\}$$

where $a = t\psi''(y) \sim te^{\varepsilon y}$

(8) $$\left|(\frac{\partial}{\partial r})^l\phi_\beta(r)\right| \leq C_l 2^{-\beta l}.$$

Define

$$\sigma_{1,t}^\beta(y,\eta,k) = \phi_\beta\left(\frac{1}{\sqrt{a} + |t\psi'(y) + k - i\eta|}\right)\sigma_{1,t}(y,\eta,k)$$

and

$$\sigma_{2,t}^\beta(y,\eta,k) = \phi_\beta(|t\psi'(y) + k - i\eta|)\sigma_{2,t}(y,\eta,k).$$

Then $\sigma_{i,t}^\beta$ $i = 1,2$ satisfies

$$\left|(\frac{\partial}{\partial\eta})^l\sigma_{i,t}^\beta(y,\eta,K)\right| \leq C_j(2^\beta\sqrt{a})^{-1-l}$$

uniformly for $y \in (-j, -j+1)$. Also

$$(9) \qquad \left|(\frac{\partial}{\partial \eta})^l(\sigma^{\beta}_{i,t}(y,\eta,k) - \sigma_{i,t}(y,\eta,k+1))\right| \le C_j(2^{\beta}\sqrt{a})^{-2-l}.$$

Define

$$F^{\beta}_{i,t}f(y,w) = \sum \frac{1}{2\pi}\int \sigma^{\beta}_{i,t}(y,\eta,k)\pi_k \tilde{f}(\eta,.)(w)e^{iy\eta}d\eta \quad i=1,2.$$

Let's put $B_{i,t} = \sum F^{\beta}_{i,t}$ $i = 1,2$. Then it's enough to show

$$\|\sum F^{\beta}_{i,t}f\|_{L^p(L^2(S,dw)dy)} \le C\|f\|_{L^p(R^-\times S,dydw)} \quad i=1,2.$$

We begin by estimating $F^N_{1,t}$. In the case $\beta = N$, we need different estimates (It's simpler) By the choice of N such that $2^N \sim 10e^{\epsilon j}\sqrt{a} \sim 10t$, we have the following. Since $\sigma^N_{1,t}$ is supported where

$$|t\psi'(y) + k - i\eta| \ge 2^N\sqrt{a} \sim 10t,$$

we have

$$|t\psi'(y) + k - i\eta| > c(1 + |\eta| + |k|) \text{ uniformly for } y < 0.$$

Hence

$$|(\partial/\partial\eta)^j(\partial/\partial y)^m\sigma^N_{1,t}(y,\eta,k)| \le C_{j,m}(1+|\eta|+|k|)^{-1-j} \quad j=0,1,...$$

It follows that $F^N_{1,t}$ is a standard pseudodifferential opperator of order -1, (Taylor [8] p.296) and we can write $F^N_{1,t}$ as

$$F^N_{1,t}f(t,w) = \int_{S\times R} K_t((w,y),(w',y'))f(w',y')dw'dy'.$$

Then the corresponding kernel has bounds

$$|K_t((w',y')(w,y))| \le C(|w-w'| + |y-y'|)^{-n+1}.$$

Also compactness of (w, w') and $n \geq 3$ give us

$$\int |K_t((w', y'), (w, y))| dw' dy' \leq C \text{ and}$$
$$\int |K_t((w', y'), (w, y))| dw dy \leq C.$$

Now we can apply Young's inequality to get $L^p \to L^p$ boundedness i.e.

$$\|F_{1,t}^N f\|_{L^p(R^- \times S, dx)} \leq C \|f\|_{L^p(R^- \times S, dx)}.$$

We will estimate the cases $\beta < N$. Let

$$M = [-t\psi'(y) - 2^\beta \sqrt{a}], M' = [M + 2 \times 2^\beta \sqrt{a}] + 1.$$

Denote

$$T_{1,t}^\beta(y, w) g(w) = \sum F_{1,t}^\beta(y, \eta, k) \pi_k g(w).$$

Then dual $L^p \to L^2$ esimate of (7) and (9) gives

$$\left\| \left(\frac{\partial}{\partial \eta} \right)^h T_{1,t}^\beta(y, \eta) \pi_{M,N} g \right\|_{L^2(S; C^m)}$$
$$\leq C_j (2^\beta \sqrt{a})^{-1-j} (t^{\frac{n-2}{2}} (2^\beta \sqrt{a})^{n/2})^{1/2 - 1/q} \|g\|_{L^p(S; C^m)}$$

uniformly for $y \in I_j = (-j, -j+1)$ where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Define

$$K_{1,t}^\beta(y, y') = \frac{1}{2\pi} \int \left(\frac{\partial}{\partial \eta} \right)^j T_{1,t}^\beta(y, \eta) \frac{1}{(i(y - y'))^j} e^{iy\eta} d\eta.$$

Since the length of the interval in η where $T_{1,t}^\beta$ is non-zero is less than $2 \times 2^\beta \sqrt{a}$,

$$\|K_{1,t}^\beta(y, y') g\|_{L^2(S)} \leq C(1 + |2^\beta \sqrt{a}|)^{-10} (t^{\frac{n-2}{2}} (2^\beta \sqrt{a})^{n/2})^{\frac{1}{2} - \frac{1}{q}} \|g\|_{L^p(S)}$$

and

$$F_{1,t}^{\beta} f(y,w) = \int K_{1,t}^{\beta}(y, y-y') f(y',.)(w) dy'.$$

Note that

$$(10) \qquad \|(1+|2^{\beta}\sqrt{a}\ z|)^{-10}\|_{L^1(R,dz)} \le C(2^{\beta}\sqrt{a})^{-1} \text{ here } z = y - y'.$$

Here is the critical part. To find the optimal range of p and q, we need negative sign of t(i.e. for the estimate to be independent of t) i.e. we need

$$t^{-1/2}(t^{\frac{n-2}{2}} t^{n/4})^{1/2-1/q} < 1.$$

This condition gives

$$q \le \frac{6n-4}{3n-6} \quad \text{and} \quad \frac{1}{p} = 1 - \frac{1}{q} \le \frac{3n+2}{6n-4}.$$

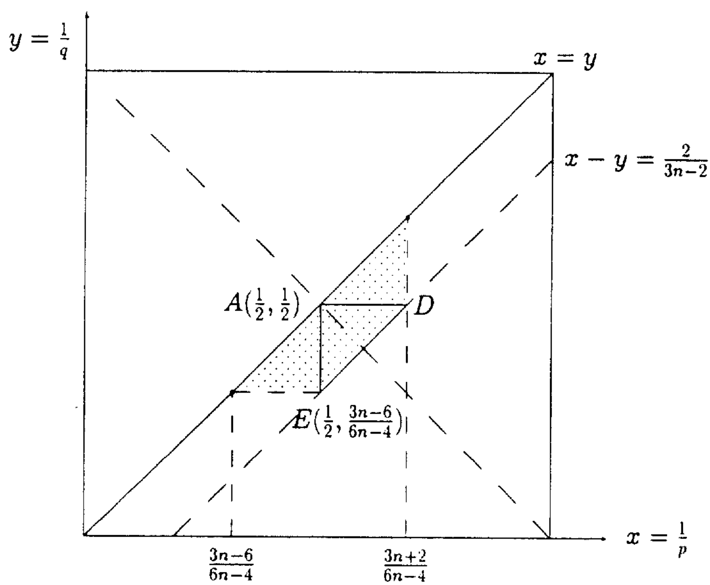

FIGURE 1. Shaded region is where (1) holds.

LEMMA 4.2. *Let $H(y,y')$ be a bounded operator from $L^2(S)$ to $L^p(S)$ of operator norm $\le h(y-y')$ for each $y, y' \in R$. Suppose that $h \in L^1(R)$. Then*

$$Tf(y,w) = \int H(y,y')f(y',.)(w)dy' \quad \textit{satisfies}$$
$$\|Tf\|_{L^p(L^2(S,dw),dy)} \le \|h\|_{L^1(R)}\|f\|_{L^p(R^- \times S,dx)}.$$

Now the lemma and (10) implies that

$$\left\|\sum^{N-1} F^{\beta}_{1,t} f\right\|_{L^p(L^2(S,dw)dy)} \le C\|f\|_{L^p(R^-\times S,dx)}. \tag{11}$$

After combining (11) and the case $\beta = N$, we get

$$\|B_{1,t}f\|_{L^p(L^2(S,dw),dy)} \le C\|f\|_{L^p(R^-\times S,dx)} \text{ for } P = \frac{6n-4}{3n+2}.$$

The same estimate is true for $B_{2,t}$ after step by step application. Now we will prove our inequality. First, define

$$P_1 f = \sum \pi_k f \quad k < -t\psi'(y)$$
$$P_2 f = \sum \pi_k f \quad k > -t\psi'(y).$$

Then $f = P_1 f + P_2 f$ and $P_1 B_1 A_t f = P_1 f$, $P_2 B_2 A_t f = P_2 f$. Now from the definition of $P_i^{'}s$, $i = 1,2$, it is easy to check that

$$\|P_i f\|_{L^p(L^2(S,dw),dy)} \le \|f\|_{L^p(L^2(S,dw),dy)} \text{ and}$$
$$\|f\|_{L^p(R^-\times S,dx)} \le C\|f\|_{L^p(L^2(S,dw),dy)}.$$

The above relations and projection method give us the desired inequality, i.e. for $p = \dfrac{6n-4}{3n+2}$

$$\begin{aligned}
\|f\|_{L^p(R^-\times S,dx)} &\le C\|f\|_{L^p(L^2(S,dw),dy)} \\
&= \|P_1 f P_2 f\|_{L^p(L^2(S,dw),dy)} \\
&\le \|P_1 f\|_{L^p(L^2(S,dw),dy)} + \|P_2 f\|_{L^p(L^2(S,dw),dy)} \\
&= \|P_1 B_1 A_t f\|_{L^p(L^2(S,dw),dy)} + \|P_2 B_2 A_t f\|_{L^p(L^2(S,dw),dy)} \\
&\le \|B_1 A_t f\|_{L^p(L^2(S,dw),dy)} + \|B_2 A_t f\|_{L^p(L^2(S,dw),dy)} \\
&\le 2C\|A_t f\|_{L^p(R^-\times S)}. \qquad (12)
\end{aligned}$$

Then by duality the following is also true

$$(13) \qquad \|f\|_{L^{p'}(R^- \times S, dx)} \leq C\|A_t f\|_{L^{p'}(R^- \times S)} \text{ for } p' = \frac{6n-4}{3n-6}.$$

Finally interpolation of (12) and (13) proves theorem.

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