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CONVEXITY PRESERVING PIECEWISE RATIONAL INTERPOLATION FOR PLANAR CURVES

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1. Introduction

A number of authors have considered the problem of shape preserving interpolation. For brevity the reader being refered to [Goodman'88, Gregory'86]. The methods in [Gregory'86] are global and non-parametric whereas [Goodman'88] discusses local and parametric shape preserving methods.

This paper uses a piecewise rational cubic interpolant to solve the problem of shape preserving interpolation for plane curves; scalar curves are also considered as a special case. The results derived here are actually the extensions of the convexity preserving results of Delbourgo and Gregory [Delbourgo and Gregory'85] who developed a C^1 shape preserving interpolation scheme for scalar curves using the same piecewise rational function. They derived the constraints, on the shape parameters occuring in the rational function under discussion, to make the interpolant preserve the convex shape of the data.

This paper begins with some preliminaries about the rational cubic interpolant. The constraints consistent with convex data, are derived in Sections 3. These constraints are dependent on the tangent vectors. The description of the tangent vectors, which are consistent and dependent on the given data, is made in Section 4. The convexity preserving results are explained with examples in Section 5.

2. The rational cubic interpolant

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Let $\mathbf{F}_i \in \mathbb{R}^2, i = 0, \cdots, n$ be a given set of data points, where $t_0 < t_1 < \cdots < t_n$. We consider the C^1 piecewise rational cubic interpolant

$$\mathbf{p}(t) = \frac{(1-\theta)^3\mathbf{F}_i + \theta(1-\theta)^2(r_i\mathbf{F}_i + h_i\mathbf{D}_i)}{1 + (r_i - 3)\,\theta(1-\theta)} + \frac{\theta^2(1-\theta)(r_i\mathbf{F}_{i+1} - h_i\mathbf{D}_{i+1}) + \theta^3\mathbf{F}_{i+1}}{1 + (r_i - 3)\,\theta(1-\theta)} \tag{1}$$

$$\theta(t) = (t - t_i)/h_i, \quad h_i = t_{i+1} - t_i.$$

We will use this to generate an interpolatory planar curve which preserves the shape of the convex data. Let

$$\begin{aligned} \mathbf{p}(t) &= (p_1(t), p_2(t)), \\ \mathbf{F}_i &= (x_i, y_i), \\ \mathbf{D}_i &= (D_i^x, D_i^y), \\ \boldsymbol{\Delta}_i &= (\Delta_i^x, \Delta_i^y), \end{aligned} \tag{2}$$

where

$$\Delta_i^x = \frac{(x_{i+1} - x_i)}{h_i}, \qquad \Delta_i^y = \frac{(y_{i+1} - y_i)}{h_i}$$

and $\mathbf{D}_i$ denote the tangent vector to the curve at the knot t_i. It can be noted that $\mathbf{p}(t)$ interpolates the points $\mathbf{F}_i$ and the tangent vectors $\mathbf{D}_i$ at the knots t_i.

The parameter r_i is to be chosen such that $r_i > -1$, which ensures a strictly positive denominator in the rational cubic. For our purposes r_i will be chosen to ensure that the interpolant preserves the shape of the data. This choice requires the knowledge of $\mathbf{p}^{(1)}(t)$ and $\mathbf{p}^{(2)}(t)$ which are as follows:

(3)

$$\mathbf{p}^{(1)}(t) = \frac{(1-\theta)^4\mathbf{D}_i + \boldsymbol{\alpha}_{1,i}\theta(1-\theta)^3 + \boldsymbol{\alpha}_{2,i}\theta^2(1-\theta)^2}{\{1 + (r_i - 3)\theta(1-\theta)\}^2} + \frac{\boldsymbol{\alpha}_{3,i}\theta^3(1-\theta) + \mathbf{D}_{i+1}\theta^4}{\{1 + (r_i - 3)\theta(1-\theta)\}^2}$$

(4)

$$\mathbf{p}^{(2)}(t) = \frac{2\{\boldsymbol{\alpha}_{4,i}(1-\theta)^3 + \boldsymbol{\alpha}_{5,i}\theta(1-\theta)^2 + \boldsymbol{\alpha}_{6,i}\theta^2(1-\theta) + \boldsymbol{\alpha}_{7,i}\theta^3\}}{h_i\{1 + (r_i - 3)\theta(1-\theta)\}^3}$$

where

$$(5)\qquad \begin{aligned}
\boldsymbol{\alpha}_{1,i} &= 2(r_i\boldsymbol{\Delta}_i - \mathbf{D}_{i+1}),\\
\boldsymbol{\alpha}_{2,i} &= (r_i^2+3)\boldsymbol{\Delta}_i - r_i(\mathbf{D}_i + \mathbf{D}_{i+1}),\\
\boldsymbol{\alpha}_{3,i} &= 2(r_i\boldsymbol{\Delta}_i - \mathbf{D}_i),\\
\boldsymbol{\alpha}_{4,i} &= r_i(\boldsymbol{\Delta}_i - \mathbf{D}_i) - \mathbf{D}_{i+1} + \mathbf{D}_i,\\
\boldsymbol{\alpha}_{5,i} &= 3(\boldsymbol{\Delta}_i - \mathbf{D}_i),\\
\boldsymbol{\alpha}_{6,1} &= 3(\mathbf{D}_{i+1} - \boldsymbol{\Delta}_i),\\
\boldsymbol{\alpha}_{7,1} &= r_i(\mathbf{D}_{i+1} - \boldsymbol{\Delta}_i) - \mathbf{D}_{i+1} + \mathbf{D}_i,
\end{aligned}$$

and we denote

$$(6)\qquad \boldsymbol{\alpha}_{j,i} = (\alpha^x_{j,i}, \alpha^y_{j,i}).$$

3. Interpolation of convex data

We assume a strictly locally convex set of data so that

$$(7a)\qquad \boldsymbol{\Delta}_i = a_i\boldsymbol{\Delta}_{i-1} + b_i\boldsymbol{\Delta}_{i+1},\ i = 1,\cdots,n-2,\ a_i, b_i > 0,$$

or equivalently, the vectors

$$(7b)\qquad \boldsymbol{\Delta}_i \times \boldsymbol{\Delta}_{i+1},\ \ i = 0,\cdots,n-2,$$

must be in the same directions. To have a convex interpolant $\mathbf{p}(t)$, and to avoid the possibility of $\mathbf{p}(t)$ having straight line segments, it is necessary that the tangent vectors should satisfy

$$(8a)\qquad \mathbf{D}_i = c_i\boldsymbol{\Delta}_{i-1} + d_i\boldsymbol{\Delta}_i,\ \ i = 0,\cdots,n-1,\ c_i, d_i > 0,$$

with appropriate end conditions $\mathbf{D}_0$ and $\mathbf{D}_n$ or equivalently the vectors

$$(8b)\qquad \mathbf{D}_i \times \boldsymbol{\Delta}_i, \boldsymbol{\Delta}_i \times \mathbf{D}_{i+1}, \boldsymbol{\Delta}_i \times \boldsymbol{\Delta}_{i+1},$$

must be in the same direction $\forall i$. Thus if

$$(9)\qquad \begin{aligned}
\beta_{1,i} &= \Delta_i^x\Delta_{i+1}^y - \Delta_i^y\Delta_{i+1}^x,\\
\beta_{2,i} &= D_i^x\Delta_i^t - D_i^y\Delta_i^x,\\
\beta_{3,i} &= \Delta_i^x D_{i+1}^y - \Delta_i^y D_{i+1}^x,\\
\beta_{4,i} &= D_i^x D_{i+1}^y - D_i^y D_{i+1}^x.
\end{aligned}$$

Then we immediately have the following:

LEMMA 1. *The conditions (7) and (8) imply that*

$$\beta_{j,i}, \quad j = 1, \cdots, 3, \quad i = 0, \cdots, n-1,$$

must be of the same sign.

Now assume, without loss of generality, that the data is consistent with a convex curve with positive curvature. Then, by Lemma 1, we must have

$$\beta_{j,i} > 0, \quad j = 1, \cdots, 3 \quad i = 0, \cdots, n-1. \tag{10}$$

Moreover $\mathbf{p}(t)$ is convex, with positive curvature, if and only if

$$p_1^{(1)}(t)\, p_2^{(2)}(t) - p_1^{(2)}(t)\, p_2^{(1)}(t) > 0, \tag{11}$$

for all $t \in [t_0, t_n]$. (The case of negative curvature can be treated in a similar way when the inequality is reversed.) After some simplifications, using (2) - (6), it can be shown that for $t \in [t_i, t_{i+1}]$,

$$p_1^{(1)}(t)\, p_2^{(2)}(t) - p_1^{(2)}(t)\, p_2^{(1)}(t) = \frac{2\sum_{j=1}^{8} \gamma_{j,i}(1-\theta)^{j-1}\theta^{8-j}}{h_i\{1+(r_i-3)\theta(1-\theta)\}^5} \tag{12}$$

where

$$\begin{aligned}
\gamma_{1,i} &= (r_i\beta_{3,i} - \beta_{4,i}),\\
\gamma_{2,i} &= 2(r_i - 1)(r_i\beta_{3,i} - \beta_{4,i}) + 3\beta_{3,i},\\
\gamma_{3,i} &= 3\beta_{2,i} + 6(r_i - 1)\beta_{3,i} + [(r_i - 1)^2 + 2](r_i\beta_{3,i} - \beta_{4,i}),\\
\gamma_{4,i} &= (7r_i - 6)\beta_{2,i} + [r_i^2 + 2(r_i - 2)^2 + r_i + 1]\beta_{3,i}\\
&\quad + (2r_i - 1)(r_i\beta_{3,i} - \beta_{4,i}),\\
\gamma_{5,i} &= (7r_i - 6)\beta_{3,i} + [r_i^2 + 2(r_i - 2)^2 + r_i + 1]\beta_{2,i}\\
&\quad + (2r_i - 1)(r_i\beta_{2,i} - \beta_{4,i}),\\
\gamma_{6,i} &= 3\beta_{3,i} + 6(r_i - 1)\beta_{2,i} + [(r_i - 1)^2 + 2](r_i\beta_{2,i} - \beta_{4,i}),\\
\gamma_{7,i} &= 2(r_i - 1)(r_i\beta_{2,i} - \beta_{4,i}) + 3\beta_{2,i},\\
\gamma_{8,i} &= (r_i\beta_{2,i} - \beta_{4,i}),
\end{aligned} \tag{13}$$

Thus, from (12), necessary conditions for convexity are

$$\gamma_{1,i} > 0 \quad \text{and} \quad \gamma_{8,i} > 0. \tag{14a}$$

Thus sufficient conditions for convexity are

$$\gamma_{j,i} > 0, \quad j = 2, \cdots, 7, \tag{14b}$$

and a sufficient condition for (14b), together with conditions (14a), is

$$r_i \geq \max\left\{1, \frac{\beta_{4,i}}{\beta_{2,i}}, \frac{\beta_{4,i}}{\beta_{3,i}}\right\}. \tag{15}$$

A number of choices of r_i can be adopted for graphical demonstration but it has been found that if

$$M_i = \max\left\{\frac{\beta_{4,i}}{\beta_{2,i}}, \frac{\beta_{4,i}}{\beta_{3,i}}\right\} \quad \text{and} \quad m_i = \min\left\{\frac{\beta_{4,i}}{\beta_{2,i}}, \frac{\beta_{4,i}}{\beta_{3,i}}\right\},$$

the choice

$$r_i = 1 + \left(1 + \frac{M_i}{2}\right)^2 + \left(1 + \frac{m_i}{2}\right)^2, \tag{16}$$

satisfies (15) and produces pleasing graphical results.

REMARK 2. (a) It follows immediately that the choice of r_i in (16) is such that $r_i > 1$.

(b) A strictly convex data set has been assumed so far. Otherwise, if $\boldsymbol{\Delta}_i = \boldsymbol{\Delta}_{i+1}$ for some i, i.e. $\mathbf{F}_i, \mathbf{F}_{i+1}$ and $\mathbf{F}_{i+2}$ are collinear, then $\mathbf{p}(t)$ must be linear on $[t_i, t_{i+2}]$. Thus we must have $\mathbf{D}_j = \mathbf{D}_{j+1} = \boldsymbol{\Delta}_j$ on $[t_j, t_{j+1}]$, $j = i, i+1$ and the rational cubic then reduces to the straight line segmet

$$\mathbf{p}(t) = (1-\theta)\mathbf{F}_j + \theta\mathbf{F}_{j+1}, \quad j = i, i+1.$$

REMARK 3. As the denominator in (1) has the form

$$(1-\theta)^3 + r_i\theta(1-\theta)^2 + r_i\theta^2(1-\theta) + \theta^3,$$

therefore (1) can be written as

$$\mathbf{p}_i(t_i) = R_0(\theta; r_i)\mathbf{F}_i + R_1(\theta; r_i)\mathbf{V}_i + R_2(\theta; r_i)\mathbf{W}_i + R_3(\theta; r_i)\mathbf{F}_{i+1},$$

where

$$\mathbf{V}_i = \mathbf{F}_i + h_i\mathbf{D}_i/r_i, \quad \mathbf{W}_i = \mathbf{F}_{i+1} - h_i\mathbf{D}_{i+1}/r_i$$

and $R_j(\theta; r_i)$, $j = 0, 1, 2, 3$, are appropriately defined rational functions with

$$\sum_{j=0}^{3} R_j(\theta; r_i) = 1.$$

Now the scalar case can be dealt with as a consequence of the identity

$$\begin{aligned} t \equiv\ & R_0(\theta; r_i)t_i + R_1(\theta; r_i)(t_i + h_i/r_i) \\ & + R_2(\theta; r_i)(t_{i+1} - h_i/r_i) + R_3(\theta; r_i)t_{i+1}, \end{aligned}$$

In fact this can be considered as an application of interpolation scheme $(t, p(t))$ in $\mathbb{R}^2$ to the values $(t_i, F_i) \in \mathbb{R}^2$ and derivatives $(1, D_i) \in \mathbb{R}^2, i = 0, \cdots, n$. It can also be noted that $\Delta_i = (1, \boldsymbol{\Delta}_i)$. Therefore the convexity constraints, in this case, are

$$r_i \geq \max\left\{\frac{D_{i+1} - D_i}{\boldsymbol{\Delta}_i - D_i}, \frac{D_{i+1} - D_i}{D_{i+1} - \boldsymbol{\Delta}_i}\right\} = 1 + m_i/M_i$$

whice are same as in [Delbourgo and Gregory'85].

4. Choice of tangent vectors

In most applications, the tangent vectors $\mathbf{D}_i$ will not be given and hence must be determined from the data $\mathbf{F}_i \in \mathbb{R}^2, i = 0, \cdots, n$. We describe here the arithmetic mean choices of tangent vectors for our plane curves which satisfy the shape preserving conditions. The arithmetic mean choice of tangent vectors is

$$\mathbf{D}_i = \lambda_i \boldsymbol{\Delta}_{i-1} + (1-\lambda_i)\boldsymbol{\Delta}_i, \quad i = 1, \cdots, n-1,$$

where

$$\lambda_i = \frac{h_i}{(h_{i-1} + h_i)},$$

while if $\mathbf{p}(t)$ is not closed, the tangents at the end points will be given as

$$\mathbf{D}_0 = \lambda_0 \boldsymbol{\Delta}_0 + (1-\lambda_0)\boldsymbol{\Delta}_{2,0},$$
$$\mathbf{D}_n = \lambda_n \boldsymbol{\Delta}_{n-1} + (1-\lambda_n)\boldsymbol{\Delta}_{n,n-2},$$

where

$$\lambda_0 = 1 + \frac{h_0}{h_1}, \quad \lambda_n = 1 + \frac{h_{n-1}}{h_{n-2}},$$
$$\boldsymbol{\Delta}_{2,0} = \frac{\mathbf{F}_2 - \mathbf{F}_0}{t_2 - t_0}, \quad \boldsymbol{\Delta}_{n,n-2} = \frac{\mathbf{F}_n - \mathbf{F}_{n-2}}{t_n - t_{n-2}}$$

These arithmetic mean approximations are suitable for convex data since they satisfy the mecessary conditions for convexity and produce pleasing graphical results.

5. Examples

The Figure 1 demonstrates the convexity preserving results corresponding to the arithmetic mean derivative values; the first and second curves in this figure represent the scalar and parametric cubic spline interpolation respectively whereas the third and the fourth curves are respectively scalar and parametric convexity preserving interpolations. The parametrization used here is the chord length parametrization. Some other parametrization can also be used.

6. Concluding remarks

C^1 rational cubic Hermite interpolant with one shape parameter has been utilized to obtain a C^1 convexity preserving plane curve method. Data dependent shape constraints are derived on the shape parameters to assure the shape preservation of the data. Choice of the tangent vectors, which are consistent and dependent on the data, has also been made. This scheme can also be implemented in the scalar case.

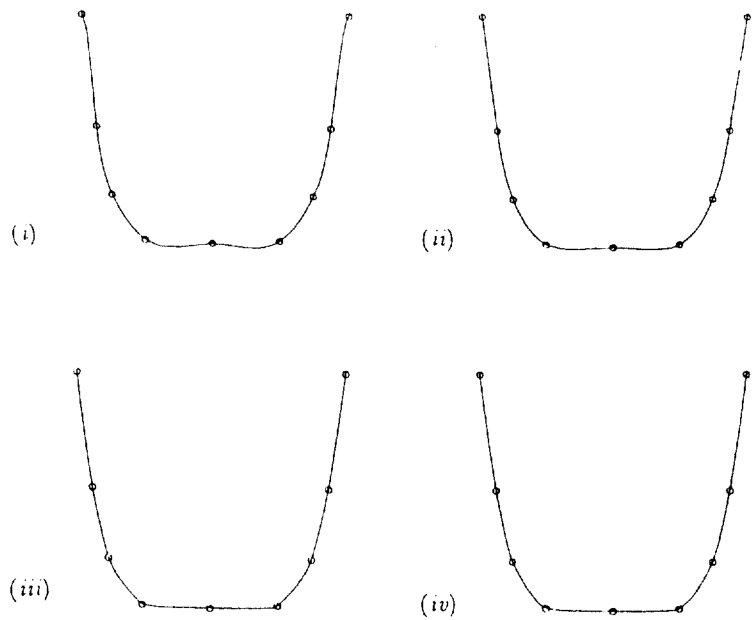

FIGURE 1. Convexity preserving rational cubic interpolation.

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