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EFFICIENT GENERATION OF MAXIMAL IDEALS IN POLYNOMIAL RINGS

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The purpose of this paper is to provide the affirmative solution of the following conjecture due to Davis and Geramita. Conjecture; Let $A = R[T]$ be a polynomial ring in one variable, where R is a regular local ring of dimension d. Then maximal ideals in A are complete intersection. Geramita has proved that the conjecture is true when R is a regular local ring of dimension 2. Bhatwadekar has proved that conjecture is true when R is a formal power series ring over a field and also when R is a localization of an affine algebra over an infinite perfect field. Nashier also proved that conjecture is true when R is a local ring of $D[X_1, \cdots, X_{d-1}]$ at the maximal ideal $(\pi, X_1, \cdots, X_{d-1})$ where $(D, (\pi))$ is a discrete valuation ring with infinite residue field.

The methods to establish our results are following from Nashier's method. We divide this paper into three sections. In section 1 we state Theorems without proofs which are used in section 2 and 3. In section 2 we prove some lemmas and propositions which are used in proving our results. In section 3 we prove our main theorem.

1.

Let R denote a commutative noetherian ring with identity. For a finitely generated R-module M over R, let $\mu(M)$ denote the least number of elements in M required to generate M as an R-module. For an ideal of a ring R, the R/I-module I/I^2 is called the conormal module of I. Since the height of an ideal $I \neq R$ is defined as the infimum of the heights of the prime divisors of I, it follows from the generalized Principal Ideal Theorem that an ideal $I \neq R$ in a noetherian ring always has finite height $ht(I) \leq \mu(I)$. If $ht(I) = \mu(I)$ we say that I is a complete intersection. More recently, it has been noticed that the properties of $\mu(I/I^2)$ and $\mu(I)$ are closely related. To be precise we have the following theorem.

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THEOREM A. *For an ideal of a noetherian ring R*

$$ht(I) \leq \mu(I/I^2) \leq \mu(I) \leq \mu(I/I^2) + 1. \ [7]$$

Which of the above theorem inequalities is an equality then becomes the question of interset. This depends very much on the types of rings and the nature of ideals therein. We shall use the following theorem, which is essentially due to Mohan Kumar[11].

THEOREM B. *Let $A = R[X]$ be a polynomial ring over a ring R of dimension d. Let I be an ideal in A containing a monic polynomial. If $n = \mu(I/I^2) \geq \dim(I) + 2$, then I is the homomorphic image of a projective R-module P of rank equal to $\mu(I/I^2)$. If in addition R is regular local then $\mu(I) = \mu(I/I^2)$.* [5]

THEOREM C. *If R is a noetherian ring and M a maximal ideal in $R[X_1, \cdots, X_n]$ and $M \cap R = P$ then P is a prime ideal of R and $ht(M) = ht(P) + n$.* [6]

THEOREM D. *If R is a noetherian ring and $M \subset R[X_1, \cdots, X_n]$ is a maximal and $M \cap R = P$ then R/P is a semilocal ring of Krull dimension ≤ 1.* [5]

2.

Let R be a ring and R_1 be a subring of R. Let $f \neq 0$ be an element in R such that f is a nonzero divisor in R. We say that $R_1 \subset R$ is an analytic isomorphism along f if $R_1/(f) \cong R/(f)$ or equivalently $R = R_1 + fR$ with $fR \cap R_1 = fR_1$. Proposition 1 is a property about an analytic isomorphism along f.

PROPOSITION 1. *Let R_1 be a regular subring of a regular local ring R. If I is an ideal in R let $I_1 = I \cap R_1$. Suppose there is an element f in I_1 such that $R_1 \subset R$ is an analytic isomorphism along f. Then;*

(1) $R_1/I_1 \cong R/I$ and $I_1/I_1^2 \cong I/I^2$
(2) $I = I_1 R$
(3) $\mu(I_1/I_1^2) = \mu(I/I^2)$

Hence $\mu(I_1) = \mu(I_1/I_1^2) \Longrightarrow \mu(I) = \mu(I/I^2)$.

Proof. [13]

DEFINITION. Let (R, m) be a local ring. We say that a monic polynomial in $R[X]$ is a Weierstrass polynomial of degree n if $f = X^n + a_1X^{n-1} + \cdots + a_n$ with $a_i \in m$ for $i = 1, 2, \cdots, n$.

LEMMA. *Let $A = R[T]$ be a polynomial ring over a ring R. Let M be a maximal ideal in A. Then $M \cap R$ is a maximal ideal in R if and only if M contains a monic polynomial.*

Proof. Let $M \cap R = P$. Suppose M contains a monic polynomial. Then it follows that $R/P \hookrightarrow A/M$ is an integral extension and hence R/P is a field as A/M is so. Conversely, if P is a maximal ideal then M/PA is a nonzero ideal in the PID $(R/P)[T]$. Now it is obvious that M contains a monic polynomial.

If (R, m) is a local ring and if $f \in R[T]$ is a monic polynomial such that (m, T) is the only maximal ideal of $R[T]$ that contains f, then the inclusion $R[T] \hookrightarrow R[T]_{(m,T)}$ is an analytic isomorphism along f.[13]

PROPOSITION 2. *Let R be a regular local ring of dimension d. Let $A = R[T]$ be a polynomial ring. Then any maximal ideal in A has height $d+1$ or d.*

Proof. Let M be a maximal ideal in A and let $M \cap R = P$. Then $\underline{M}$ is a maximal ideal in $\underline{R}[T]$, where $\underline{M} = M/PA$ and $\underline{R} = R/P$. As $\underline{M} \cap \underline{R} = (0)$, by Theorem D, $\dim \underline{R} \leq 1$. This implies that either P is the maximal ideal of R or P is a prime ideal in R of height $d-1(\dim \underline{R} + ht\ P = \dim R$ as R is regular local), and accordingly $ht\ M = d+1$ or d.

3.

Now we prove the special case of our result. Bhatwadekar also proved this theorem, but we improved the proof.

THEOREM 1. *Let K be an infinite fiels and (R, m) be a regular local ring of dimension d, where R is the localization of a K-algebra of finite type. Suppose $K \hookrightarrow R/m$ is a finite separable extension. Let $A = R[X]$ be the polynomial ring in one variable and M be a maximal* *ideal in A with $ht(M) = \dim R = d = \mu(M/M^2)$. Then M is a complete intersection.*

Proof. Let $P = M \cap R$, then $\dim R/P \leq 1$. If R/P is regular then since $ht(M/PR[X]) = 1, M/PR[X]$ is a principal ideal of $R/P[X]$. Therefore by Theorem C, $\mu(M) \leq 1 + \mu(PR[X]) = 1 + \mu(P) = 1 + ht(P) = ht(M)$. Since we always have $ht(M) \leq \mu(M)$ we get the equality $\mu(M) = ht(M)$ which shows that M is complete intersection. Now we suppose that R/P is not regular. Then $\dim R/P = 1, ht(M) = ht(P) + 1 = ht(P) + \dim(R/P) = \dim R$ and $\dim R \geq 2$.

(Case 1) $\dim R = 2$ (by Geramita [3]). Then $\dim(R/P) = 1$ implies $ht(P) = 1$. Therefore $ht(M) = ht(P) + 1 = 2$. Since $R[X]$ is regular, M is locally generated by a regular sequence of length 2. Therefore $hd_{R[X]}M = 1$ where $hd_{R[X]}M$ denotes the homological dimension of the $R[X]$-module M. Since $0 \to M \to R[X] \to R[X]/M \to 0$ is an exact sequence of R-modules we have

$$\begin{aligned} Ext^1_{R[X]}(M, R[X]) &\cong Ext^2_{R[X]}(R[X]/M, R[X]) \\ &\cong Ext^2_{R[X]}(R[X]_M/MR[X]_M, R[X]_M). \end{aligned}$$

Since $R[X]_M$ is regular local of dimension 2 and $MR[X]_M$ is generated by a regular sequence of length 2 we have

$$\begin{aligned} &Ext^2_{R[X]_M}(R[X]_M/MR[X]_M, R[X]_M) \\ \cong &Hom_{R[X]_M}(R[X]_M/MR[X]_M, R[X]_M/MR[X]_M) \text{ ([6])}. \end{aligned}$$

But

$$\begin{aligned} &Hom_{R[X]_M}(R[X]_M/MR[X]_M, R[X]_M/MR[X]_M) \\ &\cong R[X]_M/MR[X]_M \cong R[X]/M, \end{aligned}$$

we get $Ext^1_{R[X]}(M, R[X])$ to be a cyclic $R[X]$- module, Therefore by Serre [15] there is an exact sequence
$0 \longrightarrow R[X] \longrightarrow P \longrightarrow M \longrightarrow 0$ with P finitely generated projective $R[X]$-module of rank 2 and by Murthy [12] P is free. Since M is a

homomorphic image of 2 generator module P, we have $\mu(M) \le \mu(P) = 2 = ht(M) \le \mu(M)$. Hence M is a complete intersection.
(Case 2) $\dim R = d \ge 3$.
Following argument of Lindel [9] we can find a regular local subring S of R such that
(1) $S = K[T_1, \cdots, T_d]_{(f(T_1), T_2, \cdots, T_d)}$ for some polynomial ring $K[T_1, \cdots, T_d]$ over K and some irreducible polynomial $f(T_1)$ in $K[T_1]$ and
(2) There is an element h in $M \cap S$ such that $S \to R$ is an analytic isomorphism along h. Therefore $S[X] \to R[X] = A$ is an analytic isomorphism along an element h in $M \cap S[X]$ where M is a maximal ideal in $R[X]$. Hence by Proposition 1, we can replace R by S and assume that $R = K[T_1, \cdots, T_d]_{(f(T_1), T_2, \cdots, T_d)}$ and $A = R[X]$. First we assume $ht(M) = d \ge 3$. Let $D = K[T_1]_{(f(T_1))}$. Then $R = D[T_2, \cdots, T_d]_{(f(T_1), T_2, \cdots, T_d)}$ and also note that $ht(D[T_2, \cdots, T_d] \cap M) = d - 1 \ge 2$. Following the proof of Proposition 1.9 of [13] we can find a polynomial g in $D[T_2, \cdots, T_d] \cap M - f(T_1)D[T_2, \cdots, T_d]$ such that (after a change of variables) if $B = D[T_3, \cdots, T_d]_{(f(T_1), T_3, \cdots, T_d)}$ then g is a Wierstrass polynomial in $B[T_2]$ and hence $B[T_2] \subset R$ is an analytic isomorphism along g. As g is a monic polynomial in $B[T_2, X] \cap M$, $B[T_2, X] \subset R[X]$ is an analytic isomorphism along g. Let $M' = M \cap B[T_2, X]$. Since $g \in M'$, M' is a maximal ideal of $B[T_2, X]$. Moreover, by Proposition 1, $M'R[T] = M, ht(M') = ht(M)$ and $\mu(M'/M'^2) = \mu(M/M^2)$. Therefore it is enough to prove that M' is complete intersection ideal of $B[T_2, X]$. By Davis-Geramita theorem [2] M' is a complete intersection and $\mu(M') = \mu(M'/M'^2)$. Therefore by Proposition 1, $\mu(M) = \mu(M/M^2)$. We are through.
Now we prove the main theorem.

THEOREM 2. *Let (D, m) be a regular local ring with infinite residue field. Let $B = D[T_1, \cdots, T_{d-1}]$ be a polynomial ring and $\overline{M}$ a maximal ideal $(m, T_1, \cdots, T_{d-1})$ of B. Let $R = B_{(m, T_1, \cdots, T_{d-1})} = B_{\overline{M}}$ and $\dim R = d$. Set $A = R[X]$. Let $a \in M - M^2$, where M is a maximal ideal of A. Then every maximal ideal in A is a complete intersection.*

Proof. By Geramita [3] we may assume that $d \ge 3$. Since M is a maximal ideal in polynomial ring with one variable over a local ring of dimension d, it follows $ht(M) = d$ or $d + 1$ by Proposition

2. If $ht(M) = d+1 > \dim R$, by Nashier [13] $\mu(M) = \mu(M/M^2) = ht(M)$. Therefore we just consider $ht(M) = d \geq 3$. By Lindel's argument [9], there is a regular local subring B_1 of $B_{\overline{M}} = R$ such that
(1) $B_{\overline{M}} = B'_{M'}[T_1]_{(m,T_1,\cdots,T_{d-1})}$, where $B' = D[T_2,\cdots,T_{d-1}], M' = (m,T_2,\cdots,T_{d-1})$.
(2) $B_1 = B'_{M'}[T_1]$.
(3) There is an element $f \in B_1 \cap aB_{\overline{M}}$ such that $B_1 \subset B_{\overline{M}}$ is an analytic isomorphism along f. Therefore $B_1[X] \subset B_{\overline{M}}[X] = R[X] = A$ is an analytic isomorphism along f. Let $\widetilde{M} = M \cap B_1[X]$. Since $f \in \widetilde{M}, \widetilde{M}$ is a maximal ideal of $B_1[X]$. Moreover $\widetilde{M}R[X] = M$ and $ht(\widetilde{M}) = ht(M)$. Therefore it is enough to prove that $\widetilde{M}$ is a complete intersection ideal of $B_1[X]$. But $B_1[X] = B'_{M'}[X,T_1] \subset B_{\overline{M}}[X]$ is an analytic isomorphism along f and f is a monic polynomial in $B'_{M'}[X,T_1] \cap M = \widetilde{M}$. By Davis-Geramita Theorem [2], $\widetilde{M}$ is a complete intersection and $\mu(\widetilde{M}) = \mu(\widetilde{M}/\widetilde{M}^2)$. Hence, by Proposition 1, $\mu(M) = \mu(M/M^2)$ and $ht(M) = \mu(M)$. Thus we complete the proof.

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