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ON THE SUFFICIENT CONDITION FOR THE LINEARIZED APPROXIMATION OF THE BÉNARD CONVECTION PROBLEM

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1. Introduction

In various viscous flow problems it has been the custom to replace the convective derivative by the ordinary partial derivative in problems for which the data are small. In this paper we consider the Bénard Convection problem with small data and compare the solution of this problem (assumed to exist) with that of the linearized system resulting from dropping the nonlinear terms in the expression for the convective derivative. The objective of the present work is to derive an estimate for the error introduced in neglecting the convective inertia terms. In fact, we derive an explicit bound for the L_2 error. Indeed, if the initial data are $O(\varepsilon)$ where $\varepsilon \ll 1$, and the Rayleigh number is sufficiently small, we show that this error is bounded by the product of a term of $O(\varepsilon^2)$ times a decaying exponential in time. The results of the present paper then give a justification for linearizing the Benard Convection problem. We remark that although our results are derived for classical solutions, extensions to appropriately defined weak solutions are obvious. Throughout this paper we will make use of a comma to denote partial differentiation and adopt the summation convention of summing over repeated indices (in a term of an expression) from one to three.

As reference to work on continuous dependence on modelling and initial data, we mention the papers of Payne and Sather [8], Ames [2] Adelson [1], Bennett [3], Payne et al. [9], and Song [11, 12, 13, 14]. Also, a similar analysis of a micropolar fluid problem backward in time (an

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ill-posed problem) was given by Payne and Straughan [10] and Payne [7].

2. The Bénard Convection Problem

Let Ω denote the spatial region between the planes $z = 0$ and $z = 1$. The equations of the problem we are interested in are (see Galdi and Straughan [5] and Song [14]).

$$u_{j,j} = 0$$

$$(2.1) \qquad \begin{aligned} u_{i,t} + u_j u_{i,j} &= -p_{,i} + R\theta\delta_{i3} + \triangle u_i, \\ \theta_{,t} + u_j \theta_{,j} &= R u_3 + \triangle\theta, \end{aligned}$$

in $\Omega \times (0, \infty)$. Here $u_i (i = 1, 2, 3), \theta$, and p are the velocity, temperature, and presure, R is the Rayleigh number, δ_{ij} is the Kronecker-delta and $\triangle$ is three-dimensional Laplace operator. We look for solution (u_i, θ) which is periodic in x and y with a period cell D and which satisfies

$$u_i = \theta = 0 \text{ on } z = 0, 1,$$

as well as the initial conditions

$$\begin{aligned} u_i(x, 0) &= \varepsilon f_i(x), \ x \in D, \\ \theta(x, 0) &= \varepsilon g(x), \ x \in D. \end{aligned}$$

For convenience in our analysis we have taken the Prandtl number to be 1.

Assuming that the constant ε is small we may expect, for smooth enough f_i and g, that a unique global solution will exist and that this solution (u_i, θ) may be well approximated by $(\varepsilon v_i, \varepsilon\phi)$ where

$$v_{j,j} = 0,$$

$$(2.2) \qquad \begin{aligned} v_{i,t} &= -q_{,i} + R\phi\delta_{i3} + \triangle v_i, \\ \phi_{,t} &= R v_3 + \triangle\phi, \end{aligned}$$

in $D \times (0, \infty)$. Here q is an unknown scalar. The boundary conditions for $(v_i,\ \phi)$ being $v_i,\ \phi = 0$ on $z = 0, 1$, and the initial conditions are

$$v_i(x,0) = f_i(x),\ x \in D,$$
$$\phi(x,0) = g(x),\ x \in D.$$

The goal of this paper is to derive a bound for $F(t)$ where

$$F(t) = \int_D \{(u_i - \varepsilon v_i)(v_i - \varepsilon u_i) + (\theta - \varepsilon\phi)^2\}\, dx. \tag{2.3}$$

To this end we now set

$$w_i = u_i - \varepsilon v_i,\ \psi = \theta - \varepsilon \phi,$$

and observe that

$$\begin{aligned}
\frac{dF}{dt} &= 2\int_D (w_i w_{i,t} + \psi\psi_{,t})\, dx \\
&= 2\int_D w_i(\Delta w_i - p_{,i} + \varepsilon q_{,i} + R\psi\delta_{13} - u_j u_{i,j})\, dx \\
&\quad + 2\int_D \psi(\Delta\psi + Rw_3 - u_j\theta_{,j})\, dx \\
&= -2\int_D w_{i,j}w_{i,j}dx - 2\int_D \psi_{,j}\psi_{,j}dx + 4R\int_D w_3\psi dx \\
&\quad - 2\int_D w_i u_j u_{i,j} dx - 2\int_D \psi u_j \theta_{,j} dx \\
&\equiv -2(I_1 + I_2) + N_1 + N_2 + N_3.
\end{aligned} \tag{2.4}$$

Here we have carried out the obvious integration by parts, made use of the differential equations as well as the boundary and periodicity conditions. The obvious definitions of I_1, I_2, N_1, N_2, and N_3 are made. In order to derive a differential inequality for F we now need to bound

N_1, N_2, and N_3. We first observe for N_1 that

$$
\begin{aligned}
N_1 &\le R\left(4\int_D w_3^2 dx + \int_D \psi^2 dx\right)\\
&\le R\left(\frac{4}{\pi^2}\int_D {w_{3,3}}^2 dx + \frac{1}{\pi^2}\int_D {\psi_{,3}}^2 dx\right)\\
&\le \frac{R}{\pi^2}\left(\int_D {w_{3,3}}^2 dx - 2\int_D w_{3,3}w_{a,a} dx + \int_D {w_{a,a}}^2 dx\right.\\
&\qquad \left. + \int_D \psi_{,j}\psi_{,j} dx\right)\\
&\le \frac{R}{\pi^2}\left(\int_D {w_{3,3}}^2 dx - 2\int_D w_{3,a}w_{a,3} dx + \int_D w_{a,b}w_{b,a} dx\right.\\
&\qquad \left. + \int_D \psi_{,j}\psi_{,j} dx\right)\\
&\le \frac{R}{\pi^2}\left(\int_D {w_{3,3}}^2 dx + \int_D w_{3,a}w_{3,a} dx + \int_D w_{a,3}w_{a,3} dx\right.\\
&\qquad \left. + \int_D w_{a,b}w_{a,b} dx + \int_D \psi_{,j}\psi_{,j} dx\right)\\
&= \frac{R}{\pi^2}\left(\int_D w_{i,j}w_{i,j} dx + \int_D \psi_{,j}\psi_{,j} dx\right)\\
&= \frac{R}{\pi^2}\left(I_1 + I_2\right),
\end{aligned} \tag{2.5}
$$

where we have summed over repeated indices (a and b) from one to two. In deriving the inequality (2.5) we have used the arithmetic-geometric mean inequality, the one dimensional Poincaré inequality (see Hardy et al. [6]), the Cauchy-Schwarz inequality, and the fact that w_i is divergence free at step three. Turning to N_2 and N_3, we rewrite N_2 and N_3 as

$$
\begin{aligned}
N_2 &= -2\varepsilon\int_D w_i(w_j - \varepsilon v_j)v_{i,j} dx,\\
N_3 &= -2\varepsilon\int_D \psi(w_j - \varepsilon v_j)\phi_{,j} dx.
\end{aligned} \tag{2.6}
$$

Using the Cauchy-Schwarz inequality, we then find

$$
\begin{aligned}
N_2 + N_3 &= -2\varepsilon\left(\int_D w_i w_j v_{i,j}dx + \int_D w_j\psi\phi_{,j}dx\right)\\
&\quad - 2\varepsilon^2\left(\int_D w_i v_j v_{i,j}dx + \int_D \psi v_j\phi_{,j}dx\right)\\
&\le 2\varepsilon\left[\left\{\int_D (w_iw_i)^2dx\right\}^{1/2} J_1^{1/2} + \left(\int_D \psi^2 w_jw_jdx\right)^{1/2} J_2^{1/2}\right]\\
&\quad + 2\varepsilon^2\left\{\left(\int_D w_iw_iv_jv_jdx\right)^{1/2} J_1^{1/2} + \left(\int_D \psi^2 v_jv_jdx\right)^{1/2} J_2^{1/2}\right\},
\end{aligned}
\tag{2.7}
$$

where we have defined

$$
J_1 = \int_D v_{i,j}v_{i,j}dx, \quad J_2 = \int_D \phi_{,j}\phi_{,j}dx.
$$

In order to complete the bounds for N_2 and N_3, we define the Sobolev constant which was extensively used in Galdi et al. [4] and Song [14].

$$
\omega = \inf_H \frac{\|\nabla\psi\|^4}{\int_D \psi^4 dx}, \tag{2.8}
$$

where $\|.\|$ denotes the $L_2(D)$ norm and H is the set of all Dirichlet integrable functions on D which vanish on $z = 0, 1$. This Sobolev constant ω is critical to the analysis of the stability, so we use a sharper Sobolev constant ω (than that derived in Galdi et al. [4]), which was computed in Song [14]. Then we may deduce that

$$
\begin{aligned}
N_2 + N_3 &\le 2\varepsilon(\omega I_1^2 + \omega I_1 I_2)^{1/2}(J_1 + J_2)^{1/2}\\
&\quad + 2\varepsilon^2\left\{\int_D (v_jv_j)^2dx\right\}^{1/4}\left[\left\{\int_D (w_iw_i)^2dx\right\}^{1/4} J_1^{1/2}\right.\\
&\quad \left. + \left(\int_D \psi^4dx\right)^{1/4} J_2^{1/2}\right]
\end{aligned}
\tag{2.9}
$$

$$\leq 2\varepsilon\omega^{1/2} I_1^{1/2}(I_1+I_2)^{1/2}(J_1+J_2)^{1/2} + 2\varepsilon^2\omega^{1/2}(I_1+I_2)^{1/2}J_1^{1/2}(J_1+J_2)^{1/2}$$
$$\leq 2\varepsilon\omega^{1/2}(I_1+I_2)(J_1+J_2)^{1/2} + 2\varepsilon^2\omega^{1/2}(I_1+I_2)^{1/2}(J_1+J_2).$$

Here we have used the following inequality

$$(2.10)\qquad ab+cd \leq (a^2+c^2)^{1/2}(b^2+d^2)^{1/2}, \quad \text{for } a,b,c,d>0,$$

The Sobolev inequality defined in (2.8), and the Cauchy-Schwarz inequality. Using the arithmetic-geometric mean inequality, we can rewrite (2.9) for arbitrary positive β (to be determined later) as

$$(2.11)\qquad N_2+N_3 \leq 2\varepsilon\omega^{1/2}(I_1+I_2)(J_1+J_2)^{1/2} + \beta(I_1+I_2) + \varepsilon^4\omega/\beta(J_1+J_2)^2.$$

Inserting (2.5) and (2.11) back into (2.4), we end up with

(2.12)
$$dF/dt \leq -2[1-R/2\pi^2 - \varepsilon\omega^{1/2}\{J_1(t)+J_2(t)\}^{1/2} - \beta/2]\{I_1(t)+I_2(t)\} + \varepsilon^4\omega/\beta\{J_1(t)+J_2(t)\}^2.$$

Finally we proceed to estimate $J_1(t)$ and $J_2(t)$ in terms of the prescribed initial data.

3. Estimation for $J_1(t)$ and $J_2(t)$

In this section we obtain an estimation for $J_1(t)$ and $J_2(t)$ by first deriving a differential inequality. To this end we form

$$(3.1)\quad \frac{d}{dt}\{J_1(t)+J_2(t)\} = -2\int_D (v_{i,t}v_{i,t}+\phi_{,t}{}^2)dx + 2R\frac{d}{dt}\int_D \phi v_3 dx,$$

where we have employed the obvious integration by parts, the differential equations, and the boundary conditions. Integrating by dropping a negative term, we see that

$$(3.2)\qquad J_1(t)+J_2(t) \leq 2R\int \phi v_3 dx + Q_0,$$

where

$$Q_0 = J_1(0) + J_2(0) - 2R \int_D g f_3 dx$$
$$= \int_D (f_{i,j} f_{i,j} + g_{,j} g_{,j}) dx - 2R \int_D g f_3 dx,$$

which is the initial data. Using the Cauchy-Schwarz inequality, the Poincaré inequality, and the fact that

$$\int_D {v_3}^2 dx \le \frac{1}{4\pi^2} \int_D v_{i,j} v_{i,j} dx,$$

as a result of (2.5), we now further rewrite (3.2) as

$$J_1(t) + J_2(t) \le Q_0 + 2R \left(\int_D \phi^2 dx \int_D {v_3}^2 dx \right)^{1/2}$$

(3.3)
$$\le Q_0 + \frac{R}{\pi^2} \left(\int_D \phi_{i,j} \phi_{i,j} dx \int_D v_{i,j} v_{i,j} dx \right)^{1/2}$$
$$\le Q_0 + \frac{R}{2\pi^2} [J_1(t) + J_2(t)],$$

Then, we must restrict the Rayleigh number so that

$$R < 2\pi^2,$$

and we have

(3.5)
$$J_1(t) + J_2(t) \le Q_0/(1 - R/2\pi^2).$$

This inequality is valid for all time provided the prescribed initial data term (Q_0) is bounded. Thus going back to (2.12), we can replace $J_1(t) + J_2(t)$ by $Q_0/(1 - R/2\pi^2)$ and provided

(3.6)
$$1 - R/2\pi^2 - \varepsilon \omega^{1/2} Q_0^{1/2} / (1 - R/2\pi^2)^{1/2} > 0,$$

we may choose

$$\beta = 1 - R/2\pi^2 - \varepsilon\omega^{1/2}Q_0^{1/2}/(1 - R/2\pi^2)^{1/2}. \tag{3.7}$$

Then the inequality (2.12) can be rewritten as

$$\frac{d}{dt}\{F\exp(\pi^2\beta t)\} \leq \frac{\varepsilon^4\omega}{\beta}\left(\frac{Q_0}{1 - R/2\pi^2}\right)^2 \exp(\pi^2\beta t).$$

An integration yields

$$F(t) \leq \frac{\varepsilon^4\omega}{\pi^2\beta^2}\frac{{Q_0}^2}{(1 - R/2\pi^2)^2}\{1 - \exp(-\pi^2\beta t)\}. \tag{3.8}$$

This gives the desired bound for F, namely

$$F(t) \leq K\varepsilon^4\{1 - \exp(-\pi^2\beta t)\}, \tag{3.9}$$

where K is a computable constant which depends on $J_1(0) + J_2(0), \omega$, and R but not on ε. Thus the order of approximation we would expect from a purely formal manipulation is shown to be correct.

On the other hand, when $R < 2\pi^2$ for very large t, we can actually obtain a sharper bound for $F(t)$ by exploiting the crude fact that

$$\int_D (w_iw_i + \psi^2)dx \leq 2\int_D (u_iu_i + \theta^2)dx + 2\varepsilon^2\int_D (v_iv_i + \phi^2)dx. \tag{3.10}$$

Thus using the similar arguments used before, we first see that

$$\begin{aligned}\frac{d}{dt}\int_D (u_iu_i + \theta^2)dx &= -2\int_D (u_{i,j}u_{i,j} + \theta_{,j}\theta_{,j})dx + 4R\int_D \theta u_3 dx \\ &\leq -2\int_D (u_{i,j}u_{i,j} + \theta_{,j}\theta_{,j})dx \\ &\quad + \frac{R}{\pi^2}\int_D (u_{i,j}u_{i,j} + \theta_{,j}\theta_{,j})dx \\ &\leq -2(1 - R/2\pi^2)\pi^2\int_D (u_iu_i + \theta^2)dx,\end{aligned}$$

where the last step is valid if $R < 2\pi^2$. An integration leads to

$$\int_D (u_i u_i + \theta^2)dx \leq \varepsilon^2 \int_D (f_i f_i + g^2)dx \exp\{-(2\pi^2 - R)t\}. \tag{3.11}$$

In a similar way we estimate

$$\int_D (v_i v_i + \phi^2)dx \leq \int_D (f_i f_i + g^2)dx \exp\{-(2\pi^2 - R)t\}. \tag{3.12}$$

Inserting (3.11) and (3.12) back into (3.10) yields

$$\begin{aligned} 2\int_D (u_i u_i + \theta^2)dx + 2\varepsilon^2 \int_D (v_i v_i + \phi^2)dx \\ \leq 4\varepsilon^2 \int_D (f_i f_i + g^2)dx \quad \exp\{-(2\pi^2 - R)t\}. \end{aligned}$$

It follows then that this crude procedure yields, for a computable K_1, a bound of the form

$$F(t) \leq K_1 \varepsilon^2 \exp\{-(2\pi^2 - R)t\}, \tag{3.13}$$

which is of course ultraconservative for small t. Nevertheless, when $\beta < 2\pi^2 - R$, the bound (3.13) will actually be sharper than (3.9) for sufficiently large t.

4. Conclusions

In this paper we have shown that, if the Rayleigh number hypothesis (3.6) is satisfied (This restriction on R is equivalent to the weaker condition $R < 2\pi^2$, since $\varepsilon \ll 1$), we may effectively approximate the solution (u_i, θ) of the Bénard Convection problem by the solution $(\varepsilon v_i, \varepsilon \phi)$ of the analogous linear problem. The L_2 error in this approximation is seen to be of the order one would expect from a purely formal manipulation.

If $R \geq 2\pi^2$ and ε is sufficiently small, it is still possible to compare (u_i, θ) to $(\varepsilon v_i, \varepsilon\phi)$ (assuming these solutions exist).
However, in this case one would obtain an estimate of the form

$$F(t) \leq K_2 \varepsilon^4 \{\exp(\gamma t) - \exp(\gamma_1 t)\}, \ \gamma > \gamma_1,$$

for computable positive constants K_2, γ, and γ_1. This is clear if we use instead of the bound (2.5) for N_1

$$N_1 \leq 2RF.$$

Then a suitable choice for β would lead (for a computable K_3) to an inequality of the form

$$\frac{dF}{dt} \leq \gamma F + K_3 \varepsilon^4 \exp(\gamma_1 t),$$

which integrates to give the indicated bound.

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