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ON THE DIMENSION OF AMALGAMATED ORDERED SETS

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The dimension problem has been one of central themes in the theory of ordered sets. In this paper we focus on amalgamated ordered sets. Although some results can be obviously applied to infinite cases, we assume throughout that all ordered set are finite.

If A and B are ordered sets whose orders agree on $A \cap B$, then the *amalgam* of A and B is defined to be the set $A \cup B$ in which the order is the transitive closure of the union of the two orders, i.e., the smallest order containing the two orders, and is denoted by $A \vee B$. We then may have a naive conjecture that $\dim A \vee B \leq \dim A + \dim B$ for any ordered sets A and B. But it is quite surprising that the dimension of the amalgam of certain 2-dimensional ordered sets can be arbitrarily large. In fact, we have two interesting examples.

EXAMPLES. 1) Let S_n be the n-dimensional standard ordered set and U_n the ordered set obtained from S_n by subdividing every edge. Although U_n is of dimension n (see Lee, Liu, Nowakowski and Rival [3]), it is the amalgam of two 2-dimensional subsets A and B, where A consists of the maximal elements and the new ones and B consists of the minimal elements and the new ones. This was pointed out by H. A. Kierstead.

2) Let us consider the following particular subset T_n of the lattice of all subsets of $X = \{1, 2, \cdots, n\}$ which can be found in Lee, Liu, Nowakowski and Rival [3]:

$$T_n = \{\{1\}, \{2\}, \cdots, \{n\}, \{1,2\}, \{2,3\}, \cdots, \{n,1\}, \\ \cdots, X - \{n\}, X - \{1\}, \cdots, X - \{n-1\}\}.$$

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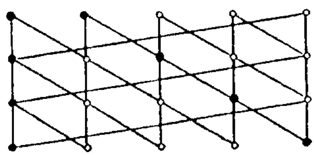
Clearly, all minimal elements and all maximal elements of T_n constitute the n-dimensional standard ordered set so that T_n is of dimension n. Now we divide T_n into two 2-dimensional subsets A and B whose amalgam is T_n itself :

$$\begin{aligned} A = \{&\{1\},\{2\},\cdots,\{n\},\{1,2\},\{2,3\},\cdots,\{n-1,n\},\\ &\cdots,X-\{n\},X-\{1\}\},\\ B = \{&\{n\},\{1\},\{n-1,n\},\{n,1\},\{1,2\},\cdots,X-\{1\},X-\{2\},\\ &\cdots,X-\{n\}\}. \end{aligned}$$

The following figure for $n = 5$ illustrates this example, where dots correspond to $A \cap B$.



But not every ordered set is divided into two 2-dimensional subsets. For instance, it is not difficult to check that S_5 cannot be the amalgam of any two of its 2-dimensional subsets. On the other hand, we observe that $A \cap B$ is an antichain in the first example and the width of $A \cap B$ in the second one is 2. Only the case that the width of $A \cap B$ is 1, i.e., $A \cap B$ is a chain, will be considered later.

Now it is natural to ask when the conjecture holds. The following is the first result on this problem.

THEOREM 1. *(Lee [2]). Let $A \vee B$ be the amalgam of ordered sets A and B. If the join (dually, the meet) of any subset of $A \cap B$ exists and belongs to $A \cap B$, then* $\dim A \vee B \leq \dim A + \dim B$.

We here mention another corollary of the preceding theorem.

COROLLARY 2. *If a lattice L is the amalgam of two sublattices A and B of L and if $A \cap B$ is also a sublattice of L, then* $\dim L \leq \dim A + \dim B$.

The theorem also tells us that if $A \cap B$ is a chain then $\dim A \vee B \leq \dim A + \dim B$. But this can be improved. For an ordered set P and

$c \in P$, we write $[c\rangle = \{x \in P | c \le x\}$, $\langle c] = \{x \in P | x \le c\}$, $(c\rangle = \{x \in P | c < x\}$ and $\langle c) = \{x \in P | x < c\}$. The *ordinal sum* $P_1 \oplus P_2 \oplus \cdots \oplus P_n$ of ordered sets $P_1, P_2, \cdots, P_n$ is the set preserving the order of each P_i with the new order relations defined by $x < y$ whenever $x \in P_j$ and $y \in P_k$ with $j < k$.

THEOREM 3. *If A and B are ordered sets such that $A \cap B$ is a chain, then* $\dim A \vee B \le \max\{\dim A, \dim B\} + 2$.

Proof. Let $A \cap B = \{c_1 < c_2 < \cdots < c_k\}$. We may assume, without loss of generality, that $m = \dim A \ge \dim B$. Let $\{E_i | i = 1, 2, \cdots, m\}$ and $\{F_i | i = 1, 2, \cdots, m\}$ be realizers of A and B, respectively. Now we construct two particular linear extensions of $A \vee B$. Let L_A be a linear extension of

$$A - [c_1\rangle \oplus \langle c_1] \cap B \oplus ((c_1\rangle - [c_2\rangle) \cap A \oplus (\langle c_2] - \langle c_1]) \cap B \oplus \cdots$$
$$\oplus((c_{k-1}\rangle - [c_k\rangle) \cap A \oplus (\langle c_k] - \langle c_{k-1}]) \cap B \oplus (c_k] \cap A \oplus B - \langle c_k]$$

and, symmetrically, L_B be a linear extension of

$$B - [c_1\rangle \oplus \langle c_1] \cap A \oplus ((c_1\rangle - [c_2\rangle) \cap B \oplus (\langle c_2] - \langle c_1]) \cap A \oplus \cdots$$
$$\oplus((c_{k-1}\rangle - [c_k\rangle) \cap B \oplus (\langle c_k] - \langle c_{k-1}]) \cap A \oplus (c_k] \cap B \oplus A - \langle c_k].$$

Then we can show, as desired, that any family of linear extensions of $A \vee B$, one from each amalgam $E_i \vee F_i (i = 1, 2, \cdots m)$, together with L_A and L_B form a realizer of $A \vee B$. For instance, if a is incomparable with b for $a \in A$ and $b \in B$ in $A \vee B$ then $a < b$ in L_A and $a > b$ in L_B.

However no ordered sets A and B have been yet found such that $A \cap B$ is a chain and $\dim A \vee B = \max\{\dim A, \dim B\} + 2$. We then pose a question.

PROBLEM. Is it true that if $A \cap B$ is a chain in the amalgam $A \vee B$ of ordered sets A and B then $\dim A \vee B \le \max\{\dim A, \dim B\} + 1$?

This question is answered positvely when A and B meet at one point.

THEOREM 4. *If A and B are ordered sets such that $|A \cap B| = 1$, then* $\dim A \vee B \leq \max\{\dim A, \dim B\} + 1$.

Proof. Let $A \cap B = \{c\}$. We may again assume that $m = \dim A \geq \dim B$. Let $\{E_i | i = 1, 2, \cdots m\}$ and $\{F_i | i = 1, 2, \cdots, m\}$ be realizers of A and B, respectively. Now the following are extensions of the amalgam $A \vee B$ for its realizer :

$$B - [c\rangle \oplus \langle c] \cap A \oplus (c\rangle \cap B \oplus A - \langle c];$$
$$E_i - [c\rangle \oplus \langle c] \cap F_i \oplus E_i - \langle c] \oplus F_i - \langle c]$$

for $i = 1, 2, ..., m$.

We define a generalized amalgam

$$\bigvee(A_i | i = 1, 2, ..., n) = \bigvee(A_i | i = 1, 2, ..., n-1) \vee A_n$$

of ordered sets A_i $(i = 1, 2, ..., n)$.

THEOREM 5. *If $A_1, A_2, ..., A_n$ are ordered sets and there is a unique common element x in all A_i, then*

$$\dim \bigvee(A_i | i = 1, 2, ..., n) \leq \max\{\dim A_i | i = 1, 2, ..., n\} + 2.$$

Proof. Let $\max\{\dim A_i | i = 1, 2, \cdots, n\} = m$. then there is a realizer $\{E_{i1}, E_{i2}, ..., E_{im}\}$ for each A_i. We now construct a realizer of $\bigvee(A_i | i = 1, 2, ..., n)$ as follows:

$$\begin{aligned}
F: \quad & A_n - [x\rangle \oplus A_{n-1} - [x\rangle \oplus \cdots \oplus A_1 - [x\rangle \oplus \{x\} \oplus \\
& (x\rangle \cap A_n \oplus (x\rangle \cap A_{n-1} \oplus \cdots \oplus (x\rangle \cap A_1; \\
G: \quad & \langle x) \cap A_n \oplus \langle x) \cap A_{n-1} \oplus \cdots \oplus \langle x) \cap A_1 \oplus \{x\} \oplus \\
& A_n - \langle x] \oplus A_{n-1} - \langle x] \oplus \cdots \oplus A_1 - \langle x]; \\
H_j: \quad & \langle x) \cap E_{1j} \oplus \langle x) \cap E_{2j} \oplus \cdots \oplus \langle x) \cap E_{nj} \oplus \{x\} \oplus \\
& (x\rangle \cap E_{1j} \oplus (x\rangle \cap E_{2j} \oplus \cdots \oplus (x\rangle \cap E_{nj}
\end{aligned}$$

for $j = 1, 2, ..., m$. In fact, suppose that a and b are incomparable for $a \in A_h$ and $b \in A_k$. Then it is enough to consider the case when $h < k$. Since it does not happen that $b \leq x \leq a$, we have $a < b$ in some H_j. On the other hand, if $a \leq x$ then $a \in A_h - \langle x]$ so that $a > b$ in G, and if $a \leq x$ then $b \in A_k - [x\rangle$ so that $a > b$ in F.

EXAMPLE. The preceding result is the best possible. For instance, the ordered set obtained from $M_7{}^2$ by deleting 0 and 1 is of dimension 4 and, on the other hand, such bound increases as the number of elements of the intersection does when the intersection is a chain (see Kelly [1]).

However we have a special case of the preceding theorem.

THEOREM 6. *If $A_1, A_2, ...A_n$ are oredered sets and x is a unique common element in all A_i that is minimal (dually, maximal) in each A_i, then* $\dim \bigvee(A_i | i = 1, 2, ..., n) \leq \max\{\dim A_i | i = 1, 2, ..., n\} + 1$.

Proof. Let $\max\{\dim A_i | i =, 1, 2..., n\} = m$. We have m linear extensions of $\bigvee(A_i | i = 1, 2, ..., n)$ as in the proof of preceding theorem and a linear extension

$$\{x\} \oplus A_n - \{x\} \oplus A_{n-1} - \{x\} \oplus \cdots \oplus A_1 - \{x\}.$$

EXAMPLES. For each $n \geq 3$, the following ordered set with n maximal elements is of dimension 3.

Recall that in Corollary 2 we considered a lattice amalgamation. When the lattice is distributive the upper bound can be improved much further.

LEMMA 7. *Every distributive lattice D of dimension at least n contains an interval isomorphic to the Boolean lattice $\mathbf{2}^n$.*

Proof. It is well known that the map f of D to $H(J(D))$ defined by $f(a) = (a] \cap J(D)$ is an isomorphism, where $J(D)$ is the ordered set of all nonzero join-irreducible elements and $H(J(D))$ is the lattice of all hereditary subsets (order ideals) of $J(D)$. Since it is also a classical fact that $\dim D = \text{width}(J(D))$, there is an n-element antichain $\{a_1, a_2, ..., a_n\}$ in $J(D)$. Let $X = \bigcup(a_i) \cap J(D)$. Then $a_i = f^{-1}((a_i]) \succ f^{-1}(X) = x$ for $i = 1, 2, ..., n$ and $a_1, a_2, ..., a_n$ are all distinct. If $\bigvee a_i = y$ then obviously the interval $[x, y]$ is isomorphic to $\mathbf{2}^n$.

THEOREM 8. *Let a distributive lattice D be the amalgam of its sublattices A and B such that $A \cap B$ is also a sublattice of D. Then* $\dim D \leq \max\{\dim A, \dim B\} + 1$. *In particular,* $\dim D = \max\{\dim A, \dim B\}$ *if* $\dim A \neq \dim B$.

Proof. Let $m = \max\{\dim A, \dim B\}$. Suppose that $\dim D > m + 1$. By Lemma 7, D contains an interval C that is isomorphic to $\mathbf{2}^{m+2}$. If there are two distinct atoms a and b of C such that $a, b \in A - B$, then $a \vee c \succ a$ and $b \vee c \succ b$ for any atom c of C other than a and b. Hence, $a \vee c, b \vee c \in A$ so that $c = (a \vee c) \wedge (b \vee c) \in A$. Thus $C \subseteq A$, which is impossible. Consequently, C has at least $m + 1$ atoms in B, so that $\dim B \geq m + 1$, which is still a contradiction, This proves the first assertion.

Next, suppose that $\dim D > m = \dim A > \dim B$. Then, again by Lemma 7, D contains $\mathbf{2}^{m+1}$ as an interval. As above, $\dim B \geq m$, which is also a contradiction.

EXAMPLES. The above inequality is the best possible. In fact, it is easy to construct an $(n+1)$-dimensional Boolean lattice by amalgamation two n-dimensional lattices that are isomorphic to $\mathbf{2}^{n-1} \times \mathbf{3}$.

However we do not know at this moment whether or not the above theorem holds for an arbitrary lattice. This problem is stated in the following.

PROBLEM. Is it true that if a lattice L is the amalgam of its sublattices A and B such that $A \cap B$ is also a sublattice of L, then $\dim L \leq \max\{\dim A, \dim B\} + 1$?

Finally we have some simple observations.

THEOREM 9. *If A and B are ordered sets such that $x, y \in A$ or $x, y \in B$ whenever (x, y) is a critical pair in $A \vee B$, then* $\dim A \vee B \leq \dim A + \dim B$.

Proof. The hypothesis implies that if (x, y) is critical in $A \vee B$ then it is also critical in A or B. Hence, the union of two respective realizers of A and B forms a set of partial extensions of $A \vee B$ realizing it.

COROLLARY 10. *If A and B are ordered sets such that $x < y$ for any $x \in A - B$ and $y \in B - A$, then* $\dim A \vee B \leq \dim A + \dim B$.

THEOREM 11. *(cf. Corollary 3.2 [2]). If A and B are ordered sets such that $A \cap B$ is an order ideal (dually, an order filter) of both A and B, then* $\dim A \vee B \leq \dim A + \dim B$.

Proof. Let $\{E_i | i \in I\}$ and $\{F_j | j \in J\}$ be realizers of A and B, respectively. Then, for each $i \in I$, we take G_i to be a linear extension of $A \vee B$ containing E_i in which every element of B is greater than elements of A whenever possible, and similarly we take H_j for each $j \in J$. In fact, x and y are incomparable for $x \in A - B$ and $y \in B - A$, while $x < y$ in all G_i and $x > y$ in all H_j.

References

1. D. Kelly, *On the dimension of partially ordered sets*, Discrete Math. **35**(1981), 135–156.
2. J. G. Lee, *The dimension of certain partially ordered sets*, Bull. of Korean Math. Soc. **25**(1988), 133–137.
3. J. G. Lee, W. P. Liu, R. Nowakowski and I. Rival, *Dimension invariance of sbudivisions*, to appear.

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