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INVARIANCE OF DOMAIN THEOREM FOR DEMICONTINUOUS MAPPINGS OF TYPE (S_+)

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1. Introduction

Wellknown invariance of domain theorems are Brower's invariance of domain theorem for continuous mappings defined on a finite dimensional space and Schauder-Leray's invariance of domain theorem for the class of mappings $I+C$ defined on a infinite dimensional Banach space with I the identity and C compact. The two classical invariance of domain theorems were proved by applying the homotopy invariance of Brower's degree and Leray-Schauder's degree respectively.

Degree theory for some class of mappings is a useful tool for mapping theorems. And mapping theorems (or surjectivity theorems of mappings) are closely related with invariance of domain theorems for mappings.

In [4,5], Browder and Petryshyn constructed a multi-valued degree theory for A-proper mappings. From this degree Petryshyn [9] obtained some invariance of domain theorems for locally A-proper mappings.

Recently Browder [6] has developed a degree theory for demicontinuous mappings of type (S_+) from a reflexive Banach space X to its dual X^*. By applying this degree we obtain some invariance of domain theorems for demicontinuous mappings of type (S_+).

2. Preliminaries

In what follows it will always be assumed that X is a reflexive Banach space with norm $\|\ \ \|$ and its dual space X^*. We use $B(x_0,r)$ and $\overline{B}(x_0,r)$ to denote respectively the open ball and the closed ball in X

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or X^* with the center x_0 and radius $r > 0$ while $\partial B(x_0, r)$ will denote its strong boundary.

In the followings 'locally' means that a mapping satisfies some properties on a neighborhood of any point in its domain. Notations $\longrightarrow$ and $\rightharpoonup$ denote the strong and weak convergence respectively. A map $T : D(T) \subset X \longrightarrow X^*$ is continuous if for any sequence $\{x_n\}$ in $D(T)$ with $x_n \longrightarrow x \in D(T)$, we have $Tx_n \longrightarrow Tx$. We need the following definitions of mappings of various monotone types.

$[M]$ A mapping $T : D(T) \subset X \longrightarrow X^*$ is said to be monotone if for any $x, y \in D(T)$, we have

$$(Tx - Ty, x - y) \geq 0.$$

$[SM]$ A mapping $T : D(T) \subset X \longrightarrow X^*$ is said to be strongly monotone if for any $x, y \in D(T)$, we have

$$(Tx - Ty, x - y) \geq c\|x - y\|^2,$$

where c is a positive constant.

$[S\phi E]$ A mapping $T : D(T) \subset X \longrightarrow X^*$ is said to be strongly $\phi-$expansive if for any $x, y \in D(T)$, we have

$$(Tx - Ty, x - y) \geq \phi(\|x - y\|),$$

where $\phi : R^+ \longrightarrow R^+$ is strictly increasing, continuous in a neighborhood of 0 and $\phi(0) = 0$.

$[S]$ A mapping $T : D(T) \subset X \longrightarrow X^*$ is said to be of type (S) if for any sequence $\{x_n\} \subset D(T)$ with $x_n \rightharpoonup x \in X$, such that $\lim(Tx_n, x_n - x) = 0$, we have $x_n \longrightarrow x$.

$[S_+]$ A mapping $T : D(T) \subset X \longrightarrow X^*$ is said to be of type (S_+) if for any sequence $\{x_n\} \subset D(T)$ with $x_n \rightharpoonup x \in X$ and $\limsup(Tx_n, x_n - x) \leq 0$, we have $x_n \longrightarrow x$.

The duality mapping $J : X \longrightarrow 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* | (x^*, x) = \|x\|^2 = \|x^*\|^2\}.$$

Let X be a reflexive Banach space which is normed so that both X and X^* are locally uniformly convex. Then the duality mapping J

is single valued, bicontinuous, strictly monotone and of type (S_+) (see Browder [6]). Browder [6] obtained the degree theory for demicontinuous mappings of type (S_+) via Galerkin approximation processes. In this degree theory the normalized mapping is the duality mapping and the homotopies are of type (S_+). Futhermore Browder [6] showed that linear homotopy is a homotopy of type (S_+).

3. Invariance of domain theorem

By applying Browder's degree we have the following invariance of domain theorem.

THEOREM 1. *Let G be an open subset of a reflexive Banach space and $T : G \longrightarrow X^*$ be demicontinuous and locally strongly ϕ−expansive. Then $T(G)$ is open in X^*.*

Proof. We choose $r > 0$ such that T is strongly ϕ−expansive on $\overline{B}(x_0, r) \subset G$. Let $y_0 = Tx_0$. Since T is strongly ϕ−expansive, T is one-to-one and $y_0 \notin T(\partial B(x_0, r))$. And $T(\partial B(x_0, r))$ is colsed. Indeed, for any sequence $\{y_n\}$ in $T(\partial B(x_0, r))$ with $y_n \longrightarrow y$, $Tx_n = y_n$, $x_n \in \partial B(x_0, r)$, we have

$$(Tx_m - Tx_n, x_m - x_n) \geq \phi(\|x_m - x_n\|).$$

Hence $\|Tx_m - Tx_n\|\|x_m - x_n\| \geq \phi(\|x_m - x_n\|)$. Since $\{x_n\}$ is bounded and $\{Tx_n = y_n\}$ is a Cauchy sequence. Hence $x_n \longrightarrow x \in \partial B(x_0, r)$. Since T is demicontinuous, $y_n = Tx_n \rightharpoonup Tx$. Therefore $y = Tx \in T(\partial B(x_0, r))$. Since $T(\partial B(x_0, r))$ is colsed, we choose $\rho > 0$ such that $\overline{B}(t_0, \rho) \cap T(\partial B(x_0, r)) = \phi$. Since T is demicontinuous and strogly ϕ−expansive on $\overline{B}(x_0, r)$, T is demicontinuous and of type (S_+). We have a homotopy of (S_+)

$$H(t, x) = tTx + (1 - t)J(x - x_0),\ y(t) = ty_0.$$

Then $y(t) \notin H(t, \partial B(x_0, r))$ for any t in [0,1]. Indeed, on the contrary we have, for some t in [0,1], for some $x \in \partial B(x_0, r)$,

$$\begin{aligned} & ty_0 = tTx + (1 - t)J(x - x_0) \\ \Longrightarrow\ & t(Tx_0 - Tx) = (1 - t)J(x - x_0) \\ \Longrightarrow\ & t(Tx_0 - Tx, x - x_0) = (1 - t)\|x - x_0\|^2 \qquad (1) \end{aligned}$$

From (1) and $\phi-$expansiveness of T we have a contradiction. Therefore $d(H(t,\bullet), B(x_0,r), y_t)$ is constant. That is,

$$(2) \qquad d(T(\bullet), B(x_0,r), y_0) = d(J(\bullet - x_0), B(x_0,r), 0)$$

On the other hand, from a homotopy of (S_+)

$$G(t,x) = tJ(x-x_0) + (1-t)Jx, \; y(t) = tJx_0$$

we have

$$(3) \qquad d(J(\bullet - x_0), B(x_0,r), 0) = d(J\bullet, B(x_0,r), J(x_0)) = 1$$

By (2) and (3), $d(T, B(x_0,r), y_0) = 1$. Since $\overline{B}(u_0,\rho) \cap T(\partial B(x_0,r)) = \phi$, for any $y \in B(t_0,\rho)$ the path $y(t) = ty_0 + (1-t)y \notin T(\partial B(x_0,r))$. Hence

$$d(T, B(x_0,r), y_0) = d(T, B(x_0,r), y) = 1.$$

Therefore $y \in T(\overline{B}(x_0,r)) \subset T(G)$. Hence $B(y_0,\rho) \subset T(\overline{B}(x_0,r)) \subset T(G)$. The proof is completed.

COROLLARY 1. *Let X be a reflexive Banach space. If $T : D(T) = X \longrightarrow X^*$ is demicontinuous and strongly monotone, then T is a homeomorphism from X to X^*.*

Proof. Since T is strongly monotone, T is one to one and $T(X)$ is closed. By Theorem 1 $T(X)$ is open. Therefore T is onto and T is a homeomorphism.

In Hilbert space we have the following result of Minty [8] and Browder [2].

COROLLARY 2. *[2,8] Let H be a Hilbert space, $G \subset H$ be open and let $T : G \longrightarrow H$ be demicontinuous and locally strongly monotone. Then $T(G)$ is open in H.*

Proof. The proof of Corollary 2 is obvious from Theorem 1.

By applying Corollary 2 and Kirszbraun's theorem, Schönberg [10] obtained the following theorem.

Schönberg's Theorem[10, Theorem1]: Let H be a Hilbert space, $G \subset H$ be open and let $T : \overline{G} \longrightarrow H$ be demicontinuous and strongly monotone. If $K \subset H$ is connected such that $K \cap T(G) \neq \phi$ and $K \cap T(\partial G) = \phi$, then $K \subset T(G)$.

Similar results are obtained by Z.Guan[7] for demicontinuous monotone mappings defined on a closure of open bounded convex subset of a reflexive Banach space. On the other hand Browder[1] has the similar results for demicontinuous monotone mapping defined on all of X. But Browder's Theorem is for bounded closed convex subsets of a reflexive Banach space.

Now we have another following similar result in Hilbert spaces.

THEOREM 2. *Let G be a bounded open subset of a Hilbert space X and $T : \overline{G} \longrightarrow X$ be demicontinuous and monotone. If $K \subset X$ is path-connected such that $K \cap T(G) \neq \phi$ and $K \cap \overline{T(\partial G)} = \phi$, then $K \subset T(G)$.*

Proof. Without loss of generality we may assume $T(0) = 0 \in K$ and $0 \in G$. For any fixed $y \in K$ we have a path $y(t)(y(0) = 0,\ y(1) = y)$ in K. Let $T_n(x) = T(x) + \frac{1}{n}x$. Since $K \cap \overline{T(\partial G)} = \phi$ for all sufficiently large n, we have

$$y(t) \notin T_n(\partial G) \tag{4}$$

For such n, let $s = \{t \in [0,1] \mid y(t) \in T_n(G)\}$. Since T_n is strongly monotone, $T_n(G)$ is open by Theorem 1. Hence S is open. Since $0 \in S$, S is nonempty. S is closed. Indeed, if $t_m \in S$, $t_m \longrightarrow t$, then we have $y(t_m) = T_n(x_m)$, $x_m \in G$, $y(t_m) \longrightarrow y(t)$. Since T_n is strongly monotone, $\{x_m\}$ is a Cauchy sequence and $x_m \longrightarrow x \in \overline{G}$. Since T_n is demicontinuous, $T_n(x_m) = y(t_m) \rightharpoonup T_n(x)$. and $y(t) = T_n(x)$. From (4) $y(t) \in T_n(G)$. Hence S is closed. We conclude that $S = [0,1]$ and $y \in T_n(G)$ for all sufficiently large n. That is, for some z_n in G

$$y = T_n(z_n) = T(z_n) + \frac{1}{n}z_n \tag{5}$$

Since T is monotone,

$$\left(\frac{1}{n}z_n - \frac{1}{m}z_m, z_n - z_m\right) \leq 0 \tag{6}$$

Due to Crandall and Pazy [3, Lemma2.4] and (6), $z_n \longrightarrow x \in \overline{G}$. By (5) and boundedness of G we have $Tx = y, x \in G$. Hence $y \in T(G)$. Therefore $K \subset T(G)$.

In the following theorem we generalize the results of Petryshyn's invariance of domain theorem [9,Theorem5].

THEOREM 3. *If T is a demicontinuous, of type (S), locally one to one mapping of an open subset G of a reflexive Banach space X into X^*, then $T(G)$ is open in X^*.*

Proof. For any x_0 in X we choose $r > 0$ such that T is monotone and one to one on $\overline{B}(x_0, r) \subset G$. Since T is one to one, $y_0 \notin T(\partial B(x_0, r))$. Since T is demicontinuous and of type (S), it is easy to show that T is demicontinuous and of type (S_+) (see[7]). So $d(T, B(x_0, r), y_0)$ is well-defined. Moreover the image of closed subset under T is closed. Indeed, let $y_n \in T(C)$, (C is a closed subset of $\overline{B}(x_0, r)$) $y_n = Tx_n, x_n \in C \subset \overline{B}(x_0, r), y_n \longrightarrow y$. Because X is reflexive, we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x$ for some x in $\overline{B}(x_0, r)$. Since $y_{n_i} = Tx_{n_i} \longrightarrow y$ and $x_{n_i} \rightharpoonup x$,

$$\begin{aligned} &\lim(Tx_{n_i} - y, x_{n_i} - x) = 0 \\ \Longrightarrow\quad &\lim(Tx_{n_i}, x_{n_i} - x) = 0 \end{aligned}$$

Since T is of type $(S), x_{n_i} \longrightarrow x \in C$. Since T is demicontinuous, $Tx_{n_i} \rightharpoonup Tx = y$. Therefore $y \in T(C)$. Hence $T(\partial B(x_0, r))$ is closed and we choose $\rho > 0$ such that $\overline{B}(Tx_0, \rho) \cap T(\partial B(x_0, r)) = \phi$. By similar methods of proof in Theorem 1 $T(G)$ is open in X^*.

References

1. F. E. Browder, *Mapping theorems for noncompact nonlinear operators in Banach spaces,* Proc. Nat. Acad. Sci. U.S. **54**(1965), 337–342.
2. F. E. Browder, *Remarks on non-linear functional equations III,* Illinois J. Math. **9**(1965), 617–622.
3. M. Crandall & A. Pazy, *Semigroups of nonlinear contractions and dissipative sets,* J. Funct. Anal. **3**(1969), 376–418.
4. F. E. Browder & W. V. Petryshyn, *The topological degree and Galerkin Approximations for noncompact operators in Banach spaces,* Bull. Amer. Math. Soc. **74**(1968), 641–646.

5. F. E. Browder & W. V. Petryshyn, *Approximation methods and the generalized topological degree for nonlinear mappings in Banach spaces,* J. Funct. Anal. **3**(1969), 217–245.
6. F. E. Browder, *Fixed point theory and nonlinear problems,* Bull. Amer. Math. Soc. **9**(1983), 1–39.
7. Z. Guan, *On operators of monotone type in Banach spaces,* Ph.D. Dissertation, University of South Florida, USA, 1990.
8. G. J. Minty, *Monotone(nonlinear) operators in Hibert space,* Duke Math. J. **29**(1962), 1038–1041.
9. W. V. Petryshyn, *Invariance of domain theorem for locally A-proper mappings and its implications,* J. Funct. Anal. **5**(1970), 137–159.
10. R. Schönberg, *Zeros of nonlinear monotone operators in Hilbert space,* Canad. Math. Bull. **21**(1978), 213–219.

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