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TIGHT CLOSURES AND INFINITE INTEGRAL EXTENSIONS

MYUNG IN MOON AND YOUNG HYUN CHO

0. Introduction

All rings are commutative, Noetherian with identity and of prime characteristic p, unless otherwise specified.

First, we describe the definition of tight closure of an ideal and the properties about the tight closure used frequently. The technique used here for the tight closure was introduced by M. Hochster and C. Huneke [4, 5, or 6].

Using the concepts of the tight closure and its properties, we will prove that if R is a complete local domain and F-rational, then R is Cohen-Macaulay.

Next, we study the properties of R^+, the integral closure of a domain in an algebraic closure of its field of fractions. In fact, if R is a complete local domain of characteristic $p > 0$, then R^+ is Cohen-Macaulay [8]. But we do not know this fact is true or not if the characteristic of R is zero. For the special case we can show that if R is a non-Cohen-Macaulay normal domain containing the rationals $\mathbf{Q}$, then R^+ is not Cohen-Macaulay. Finally we will prove that if R is an excellent local domain of characteristic p and F-rational, then R is Cohen-Macaulay.

2. Preliminaries

DEFINITION 1.1. Let $I \subseteq R$ of characteristic p be given. Let R^0 denote the complement of the union of the minimal prime ideals of R and let $I^{[q]}$ denote the ideal $(i^q : i \in I)$, $q = p^e$, $e \in \mathrm{N}$. We say that $x \in I^*$, the *tight closure* of I, if there exists $c \in R^0$ such that $cx^q \in I^{[q]}$

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for all $q >> 0$, i.e. for all sufficiently large q of the form p^e. If $I = I^*$, we say that I is *tightly closed.*

Note that if R is a domain, then $c \in R^0$ is simply $c \neq 0$. And note that I^* is an ideal of R containing I.

DEFINITION 1.2.

(1) A Noetherian ring of characteristic p is called *weakly F-regular* if every ideal is tightly closed.
(2) If every localization of R at a multiplicative subset is weakly F-regular, we say that R is *F-regular.*
(3) A Noetherian local ring of characteristic p is called *F-rational* if every ideal generated by a system of parameters (briefly, s.o.p.) is tightly-closed.

PROPOSITION 1.3.

(1) *If R is regular, then every ideal is tightly closed* [7, Theorem 4.6].
(2) *In a Cohen-Macaulay local ring R, if some s.o.p. ideal is tightly closed, then R is F-rational* [3, Proposition 2.2]

LEMMA 1.4 ([14, Lemma 2.5]). *If $(R, \underline{m})$ is a F-rational local ring, then R is normal and every part of an s.o.p. ideal is tightly closed.*

On the study of the tight closure, we find an important fact: If R is a homomorphic image of a Cohen-Macaulay ring and is a F-regular ring, then R is Cohen-Macaulay [4, 5, or 6].

Now, we raise the following question.

QUESTION. Is a complete local domain of characteristic $p > 0$ Cohen-Macaulay ?

In general, the answer is no. If so, under what condition is R Cohen-Macaulay ?

If we apply the following lemmas, then we can get a partial answer of the previous question.

LEMMA 1.5 ([6, Theorem 3.3]). *Let $R = S/I$ be an equidimensional ring of characteristic p where S is a Cohen-Macaulay ring. Let*

$x_1, \cdots, x_n$ be elements of R which are a part of an s.o.p. in R_P for all primes P which contain them. Let $J = (x_1, \cdots, x_{n-1})R$. Then $J :_R x_n \subset J^*$.

LEMMA 1.6 (THE COHEN'S STRUCTURE THEOREM).

(1) *Any complete Noetherian local ring is a homomorphic image of a complete regular local ring.*
(2) *Any complete Noetherian local domain is a finite integral extension of a regular local ring.*

THEOREM 1.7. *Let R be a complete local ring. If R is F-rational, then R is Cohen-Macaulay.*

Proof. Since R is F-rational local ring, R is normal, and hence R is a domain. Thus R is equidimensional. Let $x_1, \cdots, x_d$ generate an s.o.p. ideal, denoted by I_d, and let $I_i = (x_1, \cdots, x_i)R$, for $i = 1, \cdots, d$. Then by Lemma 1.6 and Lemma 1.5, $I_i :_R x_{i+1} \subset I_i^*$. But $I_i = I_i^*$ for all i, by Lemma 1.4. Thus $x_1, \cdots, x_d$ form a regular sequence and hence R is Cohen-Macaulay.

COROLLARY 1.8. *If R is a homomorphic image of a Cohen-Macaulay equidimensional local ring, and if there exists an s.o.p., every part of which generates a tightly closed ideal, then R is Cohen-Macaulay.*

Proof. This is obvious by Theorem 1.7.

2. Infinite Integral Extensions and Cohen-Macaulay Rings

From now on, $(R, \underline{m})$ will denote the local domain of characteristic $p > 0$ with the maximal ideal $\underline{m}$, and its field of fractions k. And R^+ will denote the integral closure of R in an algebraic closure of k. Now we study the properties of R^+, and discuss the connection of R^+ with the tight closure.

PROPOSITION 2.1. *Let $(R, \underline{m})$ be a complete local domain of characteristic p. Then*

(1) *R^+ is not Noetherian unless R is a field, but R^+ is a local domain.*
(2) *The sum of any two prime ideals in R^+ is a prime ideal.*

Proof. (1) It is clear that R^+ is not Noetherian unless R is a field. For, if R^+ were Noetherian, then the integral closure of R in its field of fractions, denoted by $\bar{R}$, would be Noetherian as a submodule of R^+. But it is known that $\bar{R}$ is not Noetherian in this case [13].

To prove R^+ is local, it is enough to show that $(*)$ for every finite extension L of k, the integral closure S of R in L is a local ring. For, if $(*)$ is satisfied, the integral closure of R in any extension of k is a local ring. Since R is a complete local domain, the integral closure of R in L is a finite R-module for every finite extension L of k. Thus S is a finite R-module. Clearly, $S \subseteq R^+$. S is complete in the $\underline{m}S$-adic topology [1, Chap 3, Corollary 2.12.1], semi-local [1, Chap 4, Corollary 2.5.3], and $\underline{m}S$ is an ideal of definition of S. Therefore, the $\underline{m}S$-adic topology on S is equivalent to the $\underline{n}$-adic topology, where $\underline{n}$ denotes the radical of S. By Proposition 2.13.18 [1, Chap 3] applied to S, we have $S = \prod_{i=1}^{q} S_i$, where each S_i is a local ring, for $i = 1, \cdots, q$. Since S is an integral domain, $q = 1$ and S is a local ring.

(2) Let P_1, P_2 be prime ideals in R^+. Suppose that $xy \in P_1 + P_2$. Let $z = y - x$, so that $xy = x(x+z) = a + b$ with $a \in P_1, b \in P_2$. Since R^+ is integrally closed, the equation $U^2 + zU = a$ has a solution $u \in R^+$. And since $u(u+z) \in P_1$, we have $u \in P_1$ or else $u + z \in P_1$. Now $x^2 + zx = -(u^2 + zu) = b$, and hence $(x-u)(x+u+z) = b \in P_2$, so that either $x - u \in P_2$ or $x + u + z \in P_2$. Since $x = (x-u) + u = (x+u+z) - (u+z)$ and $x + z = (x-u) + (u+z) = (x+u+z) - u$, there are following four cases:

i) $u \in P_1$ and $x - u \in P_2; x = (x-u) + u \in P_1 + P_2$;
ii) $u \in P_1$ and $x+u+z \in P_2; y = x+z = (x+u+z) - u \in P_1 + P_2$;
iii) $u+z \in P_1$ and $x-u \in P_2; y = x+z = (u+z) + (x-u) \in P_1 + P_2$;
iv) $u+z \in P_1$ and $x+u+z \in P_2; x = (x+u+z) - (u+z) \in P_1 + P_2$.

We see that either $x \in P_1 + P_2$ or $y = x + z \in P_1 + P_2$, as required.

PROPOSITION 2.2 ([8. Lemma 6.5]). *Let R be an arbitrary domain.*

(1) *A domain S integral over R is isomorphic with R^+ if and only if every monic polynomial over S factors into monic linear polynomials over S.*
(2) *If U is a multiplicative system in R, $(U^{-1}R)^+ \cong U^{-1}(R^+)$. In particular, for every prime ideal P of R, $(R^+)_P \cong (R_P)^+$.*

(3) *If Q is a prime ideal of R^+ lying over a prime ideal P of R, then $R^+/Q \cong (R/P)^+$.*

DEFINITION 2.3. We say that a Noetherian ring R is *excellent* if it satisfies the following three conditions:

(1) Every finitely generated R-algebra is catenary (i.e., R is universally catenary),
(2) $R_P \to \hat{R}_P$ is regular for every prime ideal P of R, and
(3) For each finitely generated R-algebra A, the set $\{P : P \in \mathrm{Spec}A$ and A_p is regular$\}$ is open in $\mathrm{Spec}A$.

Note that a homomorphism $f : R \to S$ of Noethrian rings is *regular* if f is flat, and for every $P \in \mathrm{Spec}(R)$, and for $T = S \otimes_R k(P)$, $T \otimes_{k(P)} K$ is a regular ring for every finite extension K of $k(P) = R_P/PR_P$.

The class of excellent rings is stable under localization and base extension to finitely generated algebras. Complete local Noetherian rings are excellent. Every Noetherian ring R of prime characteristic p with $[R : R^p] < \infty$ is excellent [13]. Now, we discuss the splitting problem originated from the following question.

QUESTION. Let R be a local domain containing a field of characteristic $p > 0$. Let k be the field of fractions of R and $\bar{k}$ be an algebraic closure of k. Let R^+ be the integral closure of R in $\bar{k}$. Is R^+ Cohen-Macaulay over R ?

When R is an excellent local domain of characteristic $p > 0$, the answer for the question is yes by the following theorem.

PROPOSITION 2.4 ([8, Theorem 5.15]). *Let R be an excellent local domain. Then every sequence of elements that is a part of an s.o.p. for R is a regular sequence in R^+. Hence, every relation on such a sequence in R becomes trivial in a suitable module-finite extension domain of R contained in R^+.*

COROLLARY 2.5. *Let $(R, \underline{m})$ be a complete local domain. Let $x_1, \cdots, x_d$ be an s.o.p., where $d = \dim R$. Then $x_1, \cdots, x_d$ form a regular sequence in R^+. Moreover, R^+ is a Cohen-Macaulay R-module.*

Proof. Since a complete local ring of characteristic p is excellent, this is clear by Proposition 2.4.

LEMMA 2.6 ([8]). *If R is regular and every R-sequence is an M-sequence then M is R-flat.*

COROLLARY 2.7. *Let R be a complete regular local ring, then R^+ is flat over R.*

Proof. Since R is regular, every sequence of parameters in $\underline{m}$ is a regular sequence in R. By Corollary 2.5, this parameters is also a regular sequence in R^+. That is, every R-sequence is an R^+-sequence. Thus R^+ is an R-flat module by Lemma 2.6.

COROLLARY 2.8. *Let $A = k[[x_1, \cdots, x_d]]$ be a power series ring in d variables, where k is a field of positive prime characteristic p. Then A^+ is flat over A.*

When R contains a field of characteristic 0, then the answer for the earlier question is negative.

THEOREM 2.9. *Let R be a non-Cohen-Macaulay normal domain containing the rationals $\mathbf{Q}$, then R^+ is not Cohen-Macaulay.*

To prove the above theorem, we need the following.

LEMMA 2.10 ([11, Lemma 2]). *If R is a normal domain which contains the rationals $\mathbf{Q}$ and S is an integral extension domain of R of finite degree, then R is a direct summand of S.*

Proof of Theorem 2.9. Let $x_1, \cdots, x_k$ be a part of an s.o.p. and $\sum_{i=1}^{k} r_i x_i = 0$, where $r_k \notin (x_1, \cdots, x_{k-1})R$. If R^+ is Cohen-Macaulay, then $x_1, \cdots, x_k$ would be a regular sequence in R^+, so $r_k \in (x_1, \cdots, x_{k-1})R^+$. Hence there would be a finite extension S of R such that $r_k \in (x_1, \cdots, x_{k-1})S$. Let L be a quotient field of S. Then write $r_k = \sum_{i=1}^{k-1} s_i x_i$, $s_i \in S$. Let $\phi : S \to R$ be $\frac{Tr_{L/K}}{[L:K]}$. If we let $d = [L : K]$, then $\frac{1}{d} Tr_{L/K}$ retracts S onto R. In fact, if $r \in R$ (or K), $Tr_{L/K}(r) = dr$. On the other hand, the trace of an integral element is integral and every element of K integral over R is in R. Thus this map ϕ is well-defined, R-linear, and $\phi \mid_R = id$. This implies that R is a direct summand of S as an R-module. This says every ideal of R is contracted. Thus $r_k \in (x_1, \cdots, x_{k-1})S \cap R = (x_1, \cdots, x_{k-1})R$, a contradiction.

Hochster and Huneke proved the following:

PROPOSITION 2.11 ([9]). *Let R be a reduced Noetherian ring of characteristic p and S be a module-finite extension of R and I an ideal of R. Then $IS \cap R \subseteq I^*$.*

Now we will prove the main result.

THEOREM 2.12. *Let R be an excellent local domain. Suppose that R is F-rational. Then R is Cohen-Macaulay.*

Proof. For every module-finite extension ring S of R, every ideal I generated by a part of an s.o.p. for R is contracted from S, for $IS \cap R \subset I^* = I$ by Proposition 2.11. Let $x_1, \cdots, x_k$ be part of an s.o.p. It will suffices to show that if $rx_k = \sum_{j=1}^{k-1} r_j x_j$ then $r \in (x_1, \cdots, x_{k-1})R$. But the relation becomes trivial in R^+ by Proposition 2.4, i.e., $x_1, \cdots, x_{k-1}$ is an R^+-regular sequence. Hence there exists some module-finite extension domain S of R and this relation is also trivial. Thus, $r = \sum_{j=1}^{k-1} s_j x_j$, with $s_j \in S$. But the contractedness of a part of an s.o.p. implies that

$$r \in (x_1, \cdots, x_{k-1})S \cap R = (x_1, \cdots, x_{k-1})R.$$

Thus, $x_1, \cdots, x_k$ is a regular sequence in R.

LEMMA 2.13 ([2]). *Let R be a finitely generated K-algebra where K is a perfect field of characteristic p. Then R is F-finite, i.e., ${}^1R$ is finite R-module.*

COROLLARY 2.14. *If R is a finitely generated K-algebra where K is a perfect field of characteristic p, and if R is F-rational, then R is Cohen-Macaulay.*

Proof. The F-rationality of R implies that R is normal and thus R is reduced. Then R is F-finite and excellent [13]. Hence R is Cohen-Macaulay by Theorem 2.12.

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Department of Mathematics, Seoul National University, Seoul 151–742, Korea