Bull. Korean Math. Soc. **29** (1992), No. 1, pp. 31–40

CONVERGENCE OF THE GENERALIZED IMPLICIT EULER METHOD

DONG WON YU

0. Introduction

We introduce the generalized Runge-Kutta methods with the exponentially dominant order ω in [3], and the convergence theorems of the generalized explicit Euler method are derived in [4]. In this paper we will study the convergence of the generalized implicit Euler method.

1. Preliminaries

Let us consider the initial value problem with the exponentially dominant order ω,

$$(1.1) \qquad y' = f(t, y), \qquad 0 \le t \le T, \qquad y(0) = y_0,$$

where $y(t) \in \mathbb{R}^m, f : \mathbb{R}^{m+1} \to \mathbb{R}^m$ and T can be any fixed, positive constant, large or small. The exact solution $y(t)$ of the problem (1.1) can be approximated by the implicit Euler method (IE method)

$$(1.2) \quad y_{i+1} = y_i + hf(t_{i+1}, y_{i+1}), \qquad i = 0, 1, ..., n-1, \qquad y_0 = y(0),$$

where $nh = T$, $t_i = ih$ and y_i is an approximation obtained by the method.

Using the exponentially dominant order $\omega \in \mathbb{R}$ of the problem (1.1), the IE method (1.2) can be generalized by the function $x(t) = e^{-\omega t} y(t)$ in [2] as follows:

$$(1.3) \quad y_{i+1} = e^{\omega h} y_i + hg(t_{i+1}, y_{i+1}), \quad y_0 = y(0), \quad i = 0, 1, ..., n-1,$$

where

$$g(t, y) = f(t, y) - \omega y.$$

Received January 28, 1991.

This is called the *generalized implicit Euler method* (GIE method). If the exponentially dominant order of the problem is zero, then the GIE method reduces to the IE method.

The exact solution $y(t)$ of the problem (1.1) can be represented by the integral equation (see Lawson [2]):

$$(1.4) \qquad y(t_{i+1}) = e^{\omega h}\left\{ y(t_i) + \int_0^h e^{-\omega\eta} g(t_i+\eta, y(t_i+\eta))d\eta \right\}.$$

Througout this paper, we assume the following:

ASSUMPTION. *The initial value problem (1.1) has a unique solution $y(t)$ which has continuous derivatives through the second order on [0,T]. And the function $f(t,y)$ satisfies an one-sided Lipschitz condition for all y and for $t \in [0,T]$.*

Under this ASSUMPTION and for the given exponentially dominant order ω of the problem (1.1), we can choose positive constants $M, \overline{M}$ and real valued functions $\nu, \bar{\nu}$ such that

$$(1.5) \qquad \begin{aligned} &\|y''(t)\| \le M, && 0 \le t \le T, \\ &\|\omega^2 y(t) - 2\omega y'(t) + y''(t)\| \le \overline{M}, && 0 \le t \le T, \\ &\langle f(t,y) - f(t,z), y-z\rangle \le \nu(t)\|y-z\|^2, \\ &\langle g(t,y) - g(t,z), y-z\rangle \le \bar{\nu}(t)\|y-z\|^2, \ (\bar{\nu}(t) = \nu(t) - \omega). \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is an inner product on $\mathbb{R}^m$ and $\|\cdot\|$ is the corresponding norm. Also ν and $\bar{\nu}$ are piecewise continuous functions.

If the exponentially dominant order ω is zero, then $\overline{M}$ and $\bar{\nu}(t)$ reduce to M and $\nu(t)$ respectively.

For a further development, we need the following results.

LEMMA 1. *(Greenspan & Casulli [1]) If nembers $|E_i|$, $i = 0,1,2, \ldots, n-1$, satisfy*

$$|E_{i+1}| \le A|E_i| + B, \quad i = 0,1,2,\ldots,n-1,$$

where A and B are nonnegative constants and $A \ne 1$, then

$$|E_i| \le A^i|E_0| + \frac{A^i - 1}{A-1}B, \quad i = 1,2,\ldots,n.$$

LEMMA 2. *(Greenspan & Casulli [1]) For all number t such that $1+t \geq 0$, one has*

$$0 \leq (1+t)^\alpha \leq e^{\alpha t}, \quad \alpha > 0.$$

LEMMA 3. *If $A > 0$, and i is a positive integer, then*

$$\frac{A^i - 1}{A - 1} \leq iA^{i-1}.$$

LEMMA 4. *If $0 < A < 1$ and i is a positive integer, then*

$$\frac{A^i - 1}{A - 1} \leq i.$$

LEMMA 5. *If $0 < a < A$, $B > 0$, $a \neq B$, $A \neq B$ and i is a positive integer, then*

$$\frac{(B/A)^i - 1}{B/A - 1} \leq \frac{(B/a)^i - 1}{B/a - 1}.$$

LEMMA 6. *Under the above* ASSUMPTION, *there exists a number $\xi \in (0, h)$ such that*

$$\int_0^h e^{-\omega\tau} g(t_i + \tau, y(t_i + \tau))d\tau = he^{-\omega h} g(t_{i+1}, y(t_{i+1}))$$
$$- \frac{h^2}{2} e^{-\omega\xi}\{\omega^2 y(t_i + \xi) - 2\omega y'(t_i + \xi) + y''(t_i + \xi)\}.$$

Proof. Define

$$K(x) = \int_0^x e^{-w\tau} g(t_i + \tau, y(t_i + \tau))d\tau.$$

Since $g(t, y) = f(t, y) - wy$ and $y' = f(t, y)$, we have

$$K'(x) = e^{-wx}\{y'(t_i + x) - wy(t_i + x)\},$$
$$K''(x) = e^{-wx}\{w^2 y(t_i + x) - 2wy'(t_i + x) + y''(t_i + x)\}.$$

From Taylor's theorem, there is a number ξ between x and h such that

$$K(x) = K(h) + (x - h)K'(h) + \frac{1}{2}(x - h)^2 K''(\xi).$$

Put $x = 0$. Then the proof is complete.

2. Golbal Truncation Error Bound of the GIE Method

Let us denote *global truncation errors by* $\hat{e}_i$, i.e.

$$\hat{e}_i = y(t_i) - y_i, \qquad i = 0, 1, 2, ..., n.$$

From the equality (1.4) and Lemma 6, the global truncation error bound of the GIE method (1.3) is derived for $\xi \in (0, h)$ as follows:

$$\begin{aligned}\hat{e}_{i+1} &= e^{\omega h}\hat{e}_i + h\{g(t_{i+1}, y(t_{i+1})) - g(t_{i+1}, y_{i+1})\} \\ &- \frac{h^2}{2}e^{\omega(h-\xi)}\{\omega^2 y(t_i+\xi) - 2\omega y'(t_i+\xi) + y''(t_i+\xi)\}.\end{aligned}$$

The square norm of $\hat{e}_{i+1}$ is

$$\begin{aligned}\|\hat{e}_{i+1}\|^2 &= \langle e^{\omega h}\hat{e}_i, \hat{e}_{i+1}\rangle + \langle h\{g(t_{i+1}, y(t_{i+1})) - g(t_{i+1}, y_{i+1})\}, \hat{e}_{i+1}\rangle \\ &- \langle \frac{h^2}{2}e^{\omega(h-\xi)}\{\omega^2 y(t_i+\xi) - 2\omega y'(t_i+\xi) + y''(t_i+\xi)\}, \hat{e}_{i+1}\rangle.\end{aligned}$$

Using the inequality (1.5) and the Schwarz inequality, we have

$$\|\hat{e}_{i+1}\| \le e^{\omega h}\|\hat{e}_i\| + h\bar{\nu}\|\hat{e}_{i+1}\| + \frac{h^2}{2}e^{\omega(h-\xi)}\overline{M}, \quad 0 < \xi < h. \tag{2.1}$$

First for the positive one-sided Lipschitz constant $\bar{\nu}$ of the function $g(t, y)$, and $h \in (0, 1/\bar{\nu})$, one has the following inequality from the inequality (2.1)

$$\|\hat{e}_{i+1}\| \le \frac{e^{\omega h}}{1 - h\bar{\nu}}\|\hat{e}_i\| + \frac{e^{\omega h}}{1 - h\bar{\nu}}e^{-\omega\xi}\frac{\overline{M}h^2}{2}, \quad 0 < \xi < h.$$

Using Lemma 1, we have

$$\|\hat{e}_i\| \le \left(\frac{e^{\omega h}}{1 - h\bar{\nu}}\right)^i \|\hat{e}_0\| + \frac{(\frac{e^{\omega h}}{1-h\bar{\nu}})^i - 1}{\frac{e^{\omega h}}{1-h\bar{\nu}} - 1}\frac{e^{\omega h}}{1 - h\bar{\nu}}e^{-\omega\xi}\frac{\overline{M}h^2}{2}, \quad 0 < \xi < h.$$

But, $\hat{e}_0 = 0$, so that

$$\|\hat{e}_i\| \le \frac{(\frac{e^{\omega h}}{1-h\bar{\nu}})^i - 1}{\frac{e^{\omega h}}{1-h\bar{\nu}} - 1}\frac{e^{\omega h}}{1 - h\bar{\nu}}e^{-\omega\xi}\frac{\overline{M}h^2}{2}, \quad 0 < \xi < h. \tag{2.2}$$

Now take a real number α which is greater than $\bar{\nu}$. Then there is a positive number h_b such that

$$1 - h_b\bar{\nu} = e^{-\alpha h_b}, \quad 0 < h_b < \frac{1}{\bar{\nu}}.$$

Since the inequality $1 - h\bar{\nu} > e^{-\alpha h}$ holds for the step size $h \in (0, h_b)$, the following inequality is derived from (2.2) and Lemma 5

$$\text{(2.3)} \qquad \|\hat{e}_i\| \le \frac{e^{(\omega\alpha)ih} - 1}{e^{(\omega+\alpha)h} - 1} e^{(\omega+\alpha)h} e^{-\omega\xi} \frac{\overline{M}h^2}{2}, \quad 0 < \xi < h < h_b < \frac{1}{\bar{\nu}}.$$

Appling Lemma 3 and Lemma 4 to this inequality (2.3) we can easily verify the following theorem.

THEOREM 1. *Suppose that the one-sided Lipschitz constnant $\bar{\nu}$ of the function $g(t, y)$ is positive and α is a fixed number which is greater then $\bar{\nu}$.*

(i) *If $\omega + \alpha > 0$, then for a step size $h \in (0, h_b)$ we have the global truncation error bounds ($i = 1, 2, ..., n$) as follows:*

$$\|\hat{e}_i\| \le e^{(\omega+\alpha)T} \frac{T\overline{M}h}{2}, \qquad (\text{ when } \omega > 0),$$

$$\|\hat{e}_i\| \le e^{\alpha T} \frac{TMh}{2}, \qquad (\text{ when } \omega = 0),$$

$$\|\hat{e}_i\| \le e^{(\omega+\alpha)T} e^{-\omega/\bar{\nu}} \frac{T\overline{M}h}{2}, \qquad (\text{ when } -\alpha < \omega < 0).$$

(ii) *If $\omega + \alpha < 0$, then for a step size $h \in (0, h_b)$ we have the global truncation error bounds ($i = 1, 2, ..., n$) as follows:*

$$\|\hat{e}_i\| \le e^{-\omega/\bar{\nu}} \frac{T\overline{M}h}{2}.$$

On the other hand, for the case of the nonpositive one-sided Lipschitz constant $\bar{\nu}$ of the function $g(t, y)$, the following inequality is easily derived from the inequality (2.1)

$$\|\hat{e}_{i+1}\| \le e^{\omega h}\|\hat{e}_i\| + e^{\omega(h-\xi)} \frac{\overline{M}h^2}{2}, \quad 0 < \xi < h.$$

By Lemma 1, we have

$$\|\hat{e}_i\| \le e^{\omega h}\|\hat{e}_0\|^i + \frac{(e^{\omega h})^i - 1}{e^{\omega h} - 1} e^{\omega h} e^{-\omega\xi} \frac{\overline{M}h^2}{2}, \quad 0 < \xi < h,$$

but, $\hat{e}_0 = 0$ implies

$$\|\hat{e}_i\| \le \frac{(e^{\omega h})^i - 1}{e^{\omega h} - 1} e^{\omega h} e^{-\omega\xi} \frac{\overline{M}h^2}{2}, \quad 0 < \xi < h.$$

Appling Lemma 3 and Lemma 4 to the above inequality, we have

THEOREM 2. *Suppose that the one-sided Lipschitz constant $\bar{\nu}$ of the function $g(t,y)$ is nonpositive. Then for all step size $h < T$, we have the global truncation error bounds ($i = 1, 2, ..., n$) as follows:*

$$\|\hat{e}_i\| \le e^{\omega T} \frac{T\overline{M}h}{2}, \quad (\text{ when } \omega > 0),$$

$$\|\hat{e}_i\| \le \frac{TMh}{2}, \quad (\text{ when } \omega = 0),$$

$$\|\hat{e}_i\| \le \frac{T\overline{M}h}{2}, \quad (\text{ when } \omega < 0).$$

Using the above two theorems, the following is immediate.

THEOREM 3. *For any one-sided Lipschitz constant $\bar{\nu}$ of the function $g(t,y)$ the global truncation error bounds of the GIE method (1.3) tend to zero as $h \to 0$:*

$$\lim_{h \to 0} \|\hat{e}_i\| = 0, \qquad i = 1, 2, ..., n.$$

3. Total Bound of the GIE Method

The numbers actually obtained by the GIE method (1.3) from a computer will not be the y_i but, say, some quantities u_i. These numbers satisfy an equation of the form

$$(3.1) \quad u_{i+1} = e^{\omega h} u_i + hg(t_{i+1}, u_{i+1}) + r_{i+1}(h), \quad i = 0, 1, 2, .., n-1,$$

where $r_{i+1}(h)$ represents the *local roundoff error* introduced by the inexact evaluation of the quantity

$$e^{\omega h}u_i + hg(t_{i+1}, u_{i+1}).$$

If r_0 denotes the initial roundoff error committed in evaluating y_0, then the initial condition for (3.1) becomes

$$u_0 = y(0) + r_0.$$

Now, let us consider the *total errors* E_i which are defined by

$$E_i = y(t_i) - u_i, \qquad i = 0, 1, 2, ..., n,$$

between the exact solution $y(t_i)$ and the actual numerical solution u_i. Thus, $E_0 = r_0$. For some $\xi \in (0, h)$ the total error bound of the GIE method (1.3) is derived by the equality (1.4), (3.1) and Lemma 6 as follows:

$$\begin{aligned} E_{i+1} &= e^{\omega h}E_i + h\{g(t_{i+1}, y(t_{i+1})) - g(t_{i+1}, u_{i+1})\} \\ &\quad - \frac{h^2}{2}e^{\omega(h-\xi)}\{\omega^2 y(t_i+\xi) - 2\omega y'(t_i+\xi) + y''(t_i+\xi)\} - r_{i+1}. \end{aligned}$$

The squared norm of E_{i+1} is

$$\begin{aligned} \|E_{i+1}\|^2 &= \langle e^{\omega h}E_i, E_{i+1}\rangle + \langle h\{g(t_{i+1}, y(t_{i+1})) - g(t_{i+1}, u_{i+1})\}, E_{i+1}\rangle \\ &\quad - \left\langle \frac{h^2}{2}e^{\omega(h-\xi)}\{\omega^2 y(t_i+\xi) - 2\omega y'(t_i+\xi) + y''(t_i+\xi)\}, E_{i+1}\right\rangle \\ &\quad - \langle r_{i+1}, E_{i+1}\rangle. \end{aligned}$$

Assume also, for simplicity, that the computation is being done in such a fashion that there is a positive number R which is equal to the absolute value of the maximum possible roundoff error. Then by using the inequality (1.5) and the Cauchy-Schwartz inequality we have

$$\text{(3.2)} \quad \|E_{i+1}\| \le e^{\omega h}\|E_i\| + h\bar{\nu}\|E_{i+1}| + \frac{h^2}{2}e^{\omega(h-\xi)}\overline{M} + R, \quad 0 < \xi < h.$$

First for the case of the positive one-sided Lipschitz constant $\bar{\nu}$ of the function $g(t,y)$ and $h \in (0, 1/\bar{\nu})$ the following inequality is derived from the inequality (3.2)

$$\|E_{i+1}\| \leq \frac{e^{\omega h}}{1-h\bar{\nu}}\|E_i\| + \frac{e^{\omega h}}{1-h\bar{\nu}} e^{-\omega\xi}\frac{\overline{M}h^2}{2} + \frac{R}{1-h\bar{\nu}}, \quad 0 < \xi < h.$$

Using Lemma 1, we have

$$\|E_i\| \leq \left(\frac{e_{\omega h}}{1-h\bar{\nu}}\right)^i \|E_0\| + \frac{(\frac{e^{\omega h}}{1-h\bar{\nu}})^i - 1}{\frac{e^{\omega h}}{1-h\bar{\nu}} - 1}\frac{e^{\omega h}}{1-h\bar{\nu}}\{e^{-\omega\xi}\frac{\overline{M}h^2}{2} + e^{-\omega h}R\}.$$

Since the inequality $1 - h\bar{\nu} > e^{-\alpha h}$ holds for a fixed step size $h \in (0, h_b)$, the following inequality is derived from Lemma 5 ($0 < \xi < h < h_b < \frac{1}{\bar{\nu}}$)

$$\|E_i\| \leq e^{(\omega+\alpha)ih}\|E_0\| + \frac{e^{(\omega+\alpha)ih} - 1}{e^{(\omega+\alpha)h} - 1} e^{(\omega+\alpha)h}\{e^{-\omega\xi}\frac{\overline{M}h^2}{2} + e^{-\omega h}R\}.$$

If $\omega + \alpha > 0$, then by Lemma 3 we have

$$\|E_i\| \leq e^{(\omega+\alpha)ih}\left[\|E_0\| + i\left\{e^{-\omega\xi}\frac{\overline{M}h^2}{2} + e^{-\omega h}R\right\}\right]. \tag{3.3}$$

And if $\omega + \alpha \leq 0$, then by Lemma 4 we have

$$\|E_i\| \leq \|E_0\| + i\left\{e^{-\omega\xi}\frac{\overline{M}h^2}{2} + e^{-\omega h}R\right\}. \tag{3.4}$$

From these inequalities (3.3) and (3.4) we can easily verify the following theorem.

THEOREM 4. *Suppose that the one-sided Lipschitz constant $\bar{\nu}$ of the function $g(t,y)$ is positive and α is a fixed number which is greater than $\bar{\nu}$.*

(i) *If $\omega + \alpha > 0$, then for a step size $h \in (0, h_b)$ we have the global truncation error bounds ($i = 1, 2, \ldots n$) as follows:*

$$\|E_i\| \leq e^{(\omega+\alpha)T}\left[\|E_0\| + T\left\{\frac{\overline{M}h}{2} + \frac{R}{h}\right\}\right], \qquad (\text{ when } \omega > 0),$$

$$\|E_i\| \leq e^{\alpha T}\left[\|E_0\| + T\left\{\frac{\overline{M}h}{2} + \frac{R}{h}\right\}\right], \qquad (\text{ when } \omega = 0),$$

$$\|E_i\| \leq e^{(\omega+\alpha)T}\left[\|E_0\| + Te^{-\omega/\bar{\nu}}\left\{\frac{\overline{M}h}{2} + \frac{R}{h}\right\}\right], (\text{ when } -\alpha < \omega < 0).$$

(ii) *If* $\omega + \alpha < 0$, *then for a step size* $h \in (0, h_b)$ *we have the global truncation error bounds* ($i = 1, 2, ..., n$) *as follows:*

$$\|E_i\| \le \|E_0\| + Te^{-\omega/\bar{\nu}} \left\{ \frac{\overline{M}h}{2} + \frac{R}{h} \right\}.$$

On the other hand, for the case of the nonpositive one-sided Lipschitz constant $\bar{\nu}$ of the function $g(t, y)$, the following inequality is easily derived from the inequality (3.2).

$$\|E_{i+1}\| \le e^{\omega h}\|E_i\| + \frac{h^2}{2} e^{\omega(h-\xi)}\overline{M} + R, \qquad 0 < \xi < h.$$

By Lemma 1, we have for $\xi \in (0, h)$

$$\|E_i\| \le e^{\omega i h}\|E_0\| + \frac{e^{\omega i h} - 1}{e^{\omega h} - 1} e^{\omega h} \left\{ e^{-\omega\xi} \frac{\overline{M}h^2}{2} + Re^{-\omega h} \right\}.$$

Appling Lemma 3 and Lemma 4 to the above inequality, we can arrive at

THEOREM 5. *Suppose that the one-sided Lipschitz constant* $\bar{\nu}$ *of the function* $g(t, y)$ *is nonpositive. Then for all step size* $h < T$, *we have the global truncation error bounds* ($i = 1, 2, ..., n$) *as follows:*

$$\|E_i\| \le e^{\omega T} \left[\|E_0\| + T \left\{ \frac{\overline{M}h}{2} + \frac{R}{h} \right\} \right], \quad (\text{ when } \omega > 0),$$

$$\|E_i\| \le \|E_0\| + T \left\{ \frac{Mh}{2} + \frac{R}{h} \right\}, \quad (\text{ when } \omega = 0),$$

$$\|E_i\| \le \|E_0\| + T \left\{ \frac{\overline{M}h}{2} + \frac{R}{h} \right\}, \quad (\text{ when } \omega < 0).$$

References

1. D. Greenspan and V. Casulli, *Numerical Analysis for Applied Mathematics, Science, and Engineering,* Addison-Wesley Publishing Company, 1988.
2. J. D. Lawson, *Generalized Runge-Kutta processes for stiff initial-value problems,* J. Inst. Maths. Applics., **16**(1967), 372–380.
3. D. W. Yu, *A study on Numerical Methods for Ordinary Differential Equations,* Bull. Korean Math. Soc., **28**(1991), 207–217.
4. D. W. Yu, *Convergence of the Generalized Explicit Euler Method,* Journal of the Institute for Basic Science, Chung Ang University, **4**(1990), 17–21.

DEPARTMENT OF MATHEMATICS, CHUNG ANG UNIVERSITY, SEOUL 156-756, KOREA