Bull. Korean Math. Soc. **29** (1992), No. 1, pp. 25–29

LOCALLY PRODUCT INDEFINITE KAEHLERIAN METRICS WITH VANISHING CONFORMAL CURVATURE TENSOR FIELD

JUNG-HWAN KWON AND WON-HO SOHN

0. Introduction

In 1949, S. Bochner ([2]) has introduced " Bochner curvature tensor" on a Kaehlerian manifold analogous to the Weyl conformal curvature tensor on a Riemannian manifold. In 1990, H. Kitahara, K. Matsuo and J.S. Pak([3,4]) defined a new tensor field(: conformal curvature tensor field) on a Hermitian manifold which is conformally invariant and studied some properties of the new tensor field. In 1970, S. Tachibana and R.C. Liu([5]) studied locally product Kaehlerian metrics with vanishing Bochner curvature tensor. In 1987, R. Aiyama, J.-H. Kwon and H. Nakagawa([1]) studied several properties of indefinite Kaehlerian manifold.

The purpose of this paper is to study indefinite Kaehlerian metrics with vanishing conformal curvature tensor field. In the first section, a brief summary of the complex version of indefinite Kaehlerian manifolds is recalled and we introduce the conformal curvature tensor field on an indefinite Kaehlerian manifold. In section 2, we obtain the theorem for indefinite Kaehlerian metrics with vanishing conformal curvature tensor field.

In this paper, the indices $A, B, C, D, \dots$ run over the range $\{1, \dots, 2n\}$, the indices $a, b, c, d, \dots$ run over the range $\{1, \dots, n\}$ and $a^* = n + a$, and $i, j, k, l, \dots$ run over the range $\{1, \dots, p\}$, $u, v, x, y, \dots$ run over the range $\{p+1, \dots, n\}$, $i^* = n + i$ and $x^* = n + x$.

Received January 21, 1991.
This research was partially supported by TGRC-KOSEF.

1. Indefinite Kaehlerian manifolds

Let $M = M_s^n$ be a complex $n-$dimensional indefinite Kaehlerian manifold of index $2s$ $(0 \le s \le n)$ with metric

$$ds^2 = \sum g_{AB} dz^A dz^B, \tag{1.1}$$

where $\{z^a\}$ is a local complex coordinate and z^{a^*} is a conjugate of z^a and $g_{AB} = \varepsilon_A \delta_{AB}$, where $\varepsilon_A = \pm 1$. As the metric is Kaehlerian we have $g_{ab} = g_{a^*b^*} = 0$, $g_{ab^*} = g_{b^*a}$, and (1.1) becomes $ds^2 = 2\sum \varepsilon_a \delta_{ab} dz^a dz^{b^*}$. Moreover, the Christoffel symblos $\Gamma^A{}_{BC}$ vanish except $\Gamma^a{}_{bc} = g^{ad^*} \partial g_{bd^*} / \partial z^c$ and their conjugates. As to the curvature tensor $R^A{}_{BCD}$, only the components of the form $R^a{}_{bcd^*}, R^a{}_{bc^*d}$ and their conjugates can be different form zero, and $R^a{}_{bcd^*} = \partial \Gamma^a{}_{bc} / \partial z^{d^*}$ hold good. The Ricci tensor R_{AB} satisfies $R_{ab} = R_{a^*b^*} = 0, R_{a^*b} = R_{ba^*} = \sum \varepsilon_c R_{a^*bcc^*}$ and the scalar curvature r is also given by $r = 2\sum \varepsilon_a R_{a^*a}$.

An indefinite Kaehlerian manifold is called a *space of constant holomorphic sectional curvature* if its curvature tensor satisfies

$$R_{a^*bcd^*} = \alpha \varepsilon_b \varepsilon_c (\delta_{ac}\delta_{bd} + \delta_{ab}\delta_{cd}), \tag{1.2}$$

where $\alpha = \dfrac{r}{2n(n+1)}$ is a constant.

The conformal curvature tensor field B_0 with components B_{0,a^*bcd^*} of the indefinite Kaehlerian manifold is given by ([3,4])

(1.3)

$$\begin{aligned} B_{0,a^*bcd^*} = R_{a^*bcd^*} &- \frac{1}{n}(\varepsilon_b \delta_{bd} R_{a^*c} + \varepsilon_c \delta_{ac} R_{bd^*}) \\ &+ \frac{r(n+2)}{2n^2(n+1)} \varepsilon_b \varepsilon_c \delta_{ac} \delta_{bd} - \frac{r}{2n(n+1)} \varepsilon_b \varepsilon_c \delta_{ab} \delta_{cd}. \end{aligned}$$

First of all, for the indefinite Kaehlerian manifold M the relationship between the conformal curvature tensor field of M and space of constant holomorphic sectional curvature is investigated.

THEOREM 1.1. *Let M be an indefinite Kaehlerian manifold of complex dimension $n(n>2)$. Then the following assertions are equivalent to each other:*

(1) *M has the vanishing conformal curvature tensor field,*
(2) *M is of constant holomorphic sectional curvature.*

proof. Assume that the conformal curvature tensor field B_0 vanishes indentically, from (1.3), we have

(1.4)
$$R_{a^*bcd^*} = \frac{1}{n}(\varepsilon_b\delta_{bd}R_{a^*c} + \varepsilon_c\delta_{ac}R_{bd^*}) - \frac{r(n+2)}{2n^2(n+1)}\varepsilon_b\varepsilon_c\delta_{ac}\delta_{bd} + \frac{r}{2n(n+1)}\varepsilon_b\varepsilon_c\delta_{ab}\delta_{cd}$$

and using $R_{a^*b} = \sum\varepsilon_c R_{a^*bcc^*}$ and $n>2$, we have

$$R_{a^*b} = \frac{r}{2n}\varepsilon_b\delta_{ab}. \tag{1.5}$$

Since (1.5) represents the 1st Chern class, r is constant. Substituting (1.5) into (1.4), we have

$$R_{a^*bcd^*} = \frac{r}{2n(n+1)}\varepsilon_b\varepsilon_c(\delta_{ac}\delta_{bd} + \delta_{ab}\delta_{cd}).$$

Thus M is of constant holomorphic sectional curvature. The converse is trivial.

2. Locally product indefinite Kaehlerian metrics

Consider a Kaehlerian metric (1.1) of the form

$$ds^2 = {ds_1}^2 + {ds_2}^2, \tag{2.1}$$

where ${ds_1}^2 = 2\sum\varepsilon_i\delta_{ij}dz^i dz^{j^*}$ and ${ds_2}^2 = 2\sum\varepsilon_x\delta_{xy}dz^x dz^{y^*}$ are indefinite Kaehlerian metrics of dimension p and $n-p$, respectively. For a metric of this type, we have

$$R_{i^*jxy^*} = 0. \tag{2.2}$$

Now we assume that the conformal curvature tensor field B_0 with respect to the metric of the form (2.1) vanishes. Then from (1.3) and (2.2), it follows that

$$R_{i^*j} = \left(\frac{r(n+2)}{2n(n+1)} - \frac{r_2}{2(n-p)}\right)\varepsilon_j\delta_{ij}, \tag{2.3}$$

and

$$R_{xy^*} = \left(\frac{r(n+2)}{2n(n+1)} - \frac{r_1}{2p}\right)\varepsilon_x\delta_{xy}, \tag{2.4}$$

where r_1 and r_2 denotes the scalar curvature of ds_1 and ds_2 respectively. From (2.3) or (2.4), we have

$$\frac{r(n+2)}{n(n+1)} - \frac{r_1}{p} - \frac{r_2}{(n-p)} = 0. \tag{2.5}$$

On the other hand, (2.3) and $B_{0,i^*jkl^*} = 0$ yields

$$\begin{aligned} R_{i^*jkl^*} &= \left(\frac{r(n+2)}{2n^2(n+1)} - \frac{r_2}{n(n-p)}\right)\varepsilon_k\varepsilon_j\delta_{ik}\delta_{jl} \\ &+ \frac{r}{2n(n+1)}\varepsilon_j\varepsilon_k\delta_{ij}\delta_{kl}, \end{aligned} \tag{2.6}$$

and (2.4) and $B_{0,x^*yuv^*} = 0$ yields

$$\begin{aligned} R_{x^*yuv^*} &= \left(\frac{r(n+2)}{2n^2(n+1)} - \frac{r_1}{np}\right)\varepsilon_y\varepsilon_u\delta_{xu}\delta_{yv} \\ &+ \frac{r}{2n(n+1)}\varepsilon_y\varepsilon_u\delta_{xy}\delta_{uv}. \end{aligned} \tag{2.7}$$

From (2.6), we have

$$R_{i^*j} = \left(\frac{r(n+2+np)}{2n^2(n+1)} - \frac{r_2}{n(n-p)}\right)\varepsilon_j\delta_{ij}.$$

Thus we get

$$\frac{r(n+2+np)}{n^2(n+1)} - \frac{r_1}{p} - \frac{2r_2}{n(n-p)} = 0. \tag{2.8}$$

From (2.5), (2.8) and $r = r_1 + r_2$, we see that $r = r_1 = r_2 = 0$ when $p > 1$. Thus, by (2.6) and (2.7), we obtain $R_{i^*jkl^*} = 0$ and $R_{x^*yuv^*} = 0$. Hence we have

THEOREM 2.1. *There is no locally product indefinite Kaehlerian metrics with vanishing conformal curvature tensor field except for flat.*

Combining Theorem 1.1 and Theorem 1.2, we obtain

COROLLARY 2.2. *There is no locally product indefinite Kaehlerian metrics with constant holomorphic sectional curvature except for flat.*

References

1. Aiyama, R., Kwon, J.-H. and Nakagawa, H., *Indefinite complex submanifolds of an indefinite complex space form,* J. Ramanujan Math. Soc., **2-1**(1987), 43–67.
2. Bochner, S., *Curvature and Betti number 2,* Ann. of Math. **50** (1949), 77–93.
3. Kitahara, H., Matsuo, K. and Pak, J. S., *A conformal curvature tensor field on Hermitian manifolds,* J. Korean Mant. Soc., **27-1**(1990), 7–17.
4. Kitahara, H., Matsuo, K. and Pak, J. S., *Appendium: A conformal curvature tensor field on Hermitian manifolds,* Bull. Korean Math. Soc., **27-1**(1990), 27–30.
5. Tachibana, S. and Liu, R. C., *Note on Kaehlerian metrics with vanishing Bochner curvature tensor,* Kodai Math. Sem. Rep., **22** (1970), 313–321.

DEPARTMENT OF MATHEMATICS EDUCATION, TAEGU UNIVERSITY, TAEGU 705-033, KOREA