Bull. Korean Math. Soc. **29** (1992), No. 1, pp. 15–24

PARAMETER SPACE FOR EIGENMAPS OF FLAT 3–TORI INTO SPHERES

JOON-SIK PARK† AND WON TAE OH‡

0. Introduction

Recently, the first author and H. Urakawa [3] gave a parametrization of a range-equivalence classes of all eigenmaps of arbitrary compact homogeneous Riemannian manifold (M, g) into the standard unit sphere using the idea of do Carmo and Wallach (cf.[1]), which was applied by Tóth and D'Ambra when the isotropy representation of (M, g) is irreducible (cf.[7]). Here and throughout this paper, we denote by $\mathcal{A}_\lambda(M)$ the set of all range-equivalence classes of full eigenmaps (cf.§1) of (M, g) with constant energy density $\frac{\lambda}{2}$ and by $\beta(M)$ the set of all range-equivalence classes of full minimal isometric immersions of (M, g) into the unit spheres.

The purpose of this paper is to parametrize range-equivalence classes of all eigenmaps of flat 3-tori $T^3 = R^3/\Lambda, \Lambda = c_1e_1 + c_2e_2 + c_3e_3$, into the standard unit spheres.

In this paper, we classify $\mathcal{A}_{0\lambda}(T^3)$(cf.§1) which is contained in $\mathcal{A}_\lambda(T^3)$, and determine completely the injective eigenmaps into (S^5, can) which are belonging to $\mathcal{A}_{0\lambda}(T^3)$. Moreover, as an application, we show that the only minimally imbedded flat torus into (S^5, can) which is contained in $\mathcal{A}_{0\lambda}(T^3)$ is the generalized Clifford torus.

1. Preliminaries

1.1 Let (M, g) be an arbitrary compact homogeneous Riemannian manifold. Namely, a compact connected Lie guoup G acts transitively on M whose action is written as $M \ni p \mapsto \tau_x p \in M$, for $x \in G$, and g is a G-invariant Riemannian metric on M. Denoting by K, the isotropy

Received January 8, 1991.

This paper was supported in part by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1990.

subgroup of G at some fixed point O in M, we identify M with the coset space G/K.

Let $Spec(M,g)$ be the set of all non-zero mutually distinct eigenvalues of the Laplacian $\triangle$ of (M,g) acting on the space $C^\infty(M)$ of all real valued C^∞ functions on M. For $\lambda \in Spec(M,g)$, put $V_\lambda = \{\mu \in C^\infty(M) | \triangle\mu = \lambda\mu\}$, $\dim V_\lambda = n(\lambda)+1$. The action of G on $C^\infty(M)$ defined by $\rho(x)\mu = x \circ \mu = \mu \circ \tau_x^{-1}$, $x \in G$, $\mu \in C^\infty(M)$, preserves V_λ. We define the G-invariant inner product $(\ ,\ )$ on V_λ by

$$(\mu,\mu') = \frac{\dim(V_\lambda)}{Vol(M,g)} \int_M \mu\mu' dv_g.$$

Choose an orthonormal basis of $\{f_\lambda^i\}_{i=0}^{n(\lambda)}$ of $(V_\lambda, (\ ,\ ))$ and define a C^∞ mapping $\hat{f}_\lambda$ of M into V_λ or $R^{n(\lambda)+1}$ by

$$\hat{f}_\lambda(xK) = \sum_{i=0}^{n(\lambda)} f_\lambda^i(xK) f_\lambda^i = (f_\lambda^0(xK), \cdots, f_\lambda^{n(\lambda)}(xK)).$$

Then, $\hat{f}_\lambda$ induces a C^∞ mapping f_λ of M into the unit sphere

$$S^{n(\lambda)} = \left\{ \psi = (\psi^0, \psi^1, \cdots, \psi^{n(\lambda)}) \in R^{n(\lambda)+1} \middle| \sum_{i=0}^{n(\lambda)} (\psi^i)^2 = 1 \right\}.$$

Due to the following theorem of Eells-Takahashi (cf.[5],[6]), the mapping f_λ of (M,g) into $(S^{n(\lambda)}, can)$ is harmonic with energy density $e(f_\lambda) = \frac{\lambda}{2}$.

THEOREM A (Eells - Takahashi). *Let ι be the inclusion of S^n into R^{n+1}. A smooth mapping ϕ of a Riemannian manifold (M,g) into (S^n, can) is harmonic if and only if $\triangle\Phi^i = 2e(\phi)\Phi^i, i = 0,1,2,3,\cdots,n$, where $i \circ \phi = (\Phi^0, \Phi^1, \cdots, \Phi^n)$. Here $e(\phi)$ is the energy density of ϕ which, for an orthonormal frame field $\{\mathbf{e}_j\}_{j=1}^m$ of (M,g) $(m = \dim M)$, is by definition, $e(\phi) = \frac{1}{2}\sum_{j=1}^m (\phi^* can)(e_j, e_j)$.*

We call the above map f_λ the *standard eigenmap* of (M,g) into $(S^{n(\lambda)}, can)$ associated to the eigenvalue λ. Let W_0 denote the linear subspace of the symmetric square $S^2(V_\lambda) = S^2(R^{n(\lambda)+1})$ given by

$$W_0 = Span_R\{(\rho(a)v_0)^2 \in S^2(V_\lambda) | a \in G\},$$

where $v_0 := \hat{f}_\lambda(K) \in V_\lambda$, and set $E_\lambda = (W_0)^\perp \subset S^2(R^{n(\lambda)+1})$, where the orthogonal complement is taken with respect to the inner product $<A, B> = trace(AB)$, $A, B \in S^2(R^{n(\lambda)+1})$. Then the first author got in a joint paper (cf.[3]) with H. Urakawa

THEOREM B. *Let (M, g) be a compact homogeneous Riemannian manifold.*

(1) *If ϕ is a full eigenmap of (M, g) into (S^n, can) with energy density $\frac{\lambda}{2}$, then $\lambda \in Spec(M, g)$ and $n \leq n(\lambda)$.*
(2) *The set $\mathcal{A}_\lambda(M)$ can be parametrized by the compact convex body $L_\lambda = \{C \in E_\lambda | C + I \geq 0\}$ in the vector space E_λ. The interior points of L_λ correspond to full eigenmaps into $(S^{n(\lambda)}, can)$, and the boundary points correspond to full eigenmaps into (S^n, can), $n \leq n(\lambda)$. The correspondence is given by $L_\lambda \ni C \mapsto \sqrt{C + I} f_\lambda$.*

Here, two maps $f, f' : M \mapsto S^n$ are said to be range-equivalent if there exists $U \in O(n+1)$ such that $f' = U \circ f$. And, a map $f : M \mapsto S^n$ is said to be *full* if the image $f(M)$ is not contained in any great sphere in S^n.

1.2 In the following, we assume that (M, g) is a flat torus, i.e., $M = T^m = R^m/\Lambda$, where $\Lambda = Z\mathbf{a}_1 + Z\mathbf{a}_2 + \cdots + Z\mathbf{a}_m$ is a lattice of R^m and $g = g_\Lambda$ is induced from the standard Euclidean inner product $<,>$ of R^m.

The spectrum $Spec(R^m/\Lambda, g_\Lambda)$ of $\triangle$ of $(R^m/\Lambda, g_\Lambda)$ is given as follows:

$$\begin{aligned} the\ \ eigenvalues &= 4\pi^2 \|\mathbf{n}\|^2, & \mathbf{n} \in \Lambda^*, \\ the\ \ eigenfunctions &= e^{2\pi i \mathbf{n}\cdot\mathbf{x}}, & \mathbf{x} \in R^m. \end{aligned}$$

Note that each eigenvalue has even multiplicity. Here $\Lambda^* = Z\xi_1 \oplus \cdots \oplus Z\xi_m$, the dual lattice of Λ, i.e., $<\xi_i, \mathbf{a}_j> = \delta_{ij}$, and $\mathbf{n} \cdot \mathbf{x} = <\mathbf{n}, \mathbf{x}> = \sum_{i=1}^m n_i x_i$ and $\|\mathbf{x}\|^2 = <\mathbf{x}, \mathbf{x}>$. We denote for $\mathbf{n} \in \Lambda^*$,

$$f_{\mathbf{n}}(\mathbf{x}) := \cos 2\pi \mathbf{n} \cdot \mathbf{x}, \qquad g_{\mathbf{n}}(\mathbf{x}) := \sin 2\pi \mathbf{n} \cdot \mathbf{x},$$

and

$$V_{\mathbf{n}} := \{f_{\mathbf{n}}, g_{\mathbf{n}}\}_R \subset C^\infty(T^m).$$

Then the translations of T^m on T^m induce a T^m-action on $V_{\mathbf{n}}$ by $a(\mathbf{x}')f(\mathbf{x}) := f(\mathbf{x}-\mathbf{x}')$, for $a(\mathbf{x}') := \sum_{i=1}^{m} x_i'\mathbf{a}_i$ and $f \in V_{\mathbf{n}}$. We introduce a lexicographic order $>$ on Λ^* by setting $\xi_1 > \xi_2 > \cdots > \xi_m > 0$. Now let λ be the eigenvalue of Δ of $(R^m/\Lambda, g_\Lambda)$ with multiplicity, say $2p$. Then we take a subset $\{\mathbf{n}_j\}_{j=1}^{2p}$ of Λ^* such that $\|\mathbf{n}_j\|^2 = \lambda/4\pi^2, 1 \leq j \leq 2p$, and

$$\{\mathbf{n}_j\}_{j=1}^{p} = \{\mathbf{n} \in \Lambda^* | n \geq 0, \|\mathbf{n}\|^2 = \lambda/4\pi^2\}.$$

The eigenspace V_λ is decomposed as $V_\lambda = \sum_{j=1}^{p} \oplus V_{\mathbf{n}j}$, and the inner product $(\ ,\ )$ on V_λ is

$$\begin{aligned}(\mu, \mu') &= \frac{2p}{Vol(T^m, g_\lambda)} \int_{T^m} \mu\mu' dv_{g_\Lambda} \\ &= 2p \int_0^1 \cdots \int_0^1 \mu(\mathbf{x})\mu'(\mathbf{x}) dx_1 \cdots dx_m,\end{aligned}$$

for $\mu, \mu' \in V_\Lambda, \mathbf{x} = \sum_{i=1}^{m} x_i\mathbf{a}_i$. Then $\{f_{\mathbf{n}j}/\sqrt{p}, g_{\mathbf{n}_j}/\sqrt{p}\}_{j=1}^{p}$ is an orthonormal basis of $(V_\lambda, (\ ,\ ))$, and the standard eigenmap of (T^m, g_Λ) into (S^{2p-1}, can) is

$$\begin{aligned}f_\lambda &= \frac{1}{\sqrt{p}}(f_{\mathbf{n}_1}, g_{\mathbf{n}_1}, \cdots, f_{\mathbf{n}_p}, g_{\mathbf{n}_p}) \\ &= \frac{1}{\sqrt{p}}(\exp(2\pi i\mathbf{n}_1 \cdot \mathbf{x}), \cdots, \exp(2\pi i\mathbf{n}_p \cdot \mathbf{x})).\end{aligned}$$

Then, $\hat{f}_\lambda(0) = \sum_{i=1}^{p} \frac{1}{\sqrt{p}} f_{\mathbf{n}_i}$, and
(1.1)

$$(a(\mathbf{x}')v_0)^2 = p^{-1}\left\{\sum_{i=1}^{p}(\cos 2\pi\mathbf{n}_j \cdot \mathbf{x}' f_{\mathbf{n}_j} + \sin 2\pi\mathbf{n}_j \cdot \mathbf{x}' g_{\mathbf{n}_j})\right\}^2 \in S^2(V_\lambda).$$

Then, the first author got the following two theorems in a joint paper(cf. [3])with H. Urakawa

THEOREM C. *Let (M, g) be a flat torus $(R^m/\Lambda, g_\Lambda)$.*

(1) *Then, for each $\lambda \in Spec(M, g)$,*

$$\dim(\mathcal{A}_\lambda(M)) = \dim(E_\lambda) = 2p^2 + p - 1 - 2N \geq p - 1,$$

where N is the number of mutually distinct elements of $\{\mathbf{n}_j + \mathbf{n}_k (1 \le j \le k \le p), \mathbf{n}_j - \mathbf{n}_k (1 \le j < k \le p)\}$.

(2) *Moreover, let* $\{\mathbf{m}_j\}_{j=1}^{N}$ *be the set of mutually distinct elements in* (1). *Then the subspace* W_0 *of* $S^2(V_\lambda)$ *coincides with the* $(2N+1)$*-dimensional subspace of* $S^2(V_\lambda)$ *spanned by* $\{\mathbf{b}_0, \mathbf{b}_1, \cdots, \mathbf{b}_N, \mathbf{b}'_1, \cdots, \mathbf{b}'_N\}$, *where* $\mathbf{b}_0 = \sum_{j=1}^{p}(f_{\mathbf{n}_j}^2 + g_{\mathbf{n}_j}^2)$, *and for* $s = 1, 2, \cdots, N$,

$$\mathbf{b}_s := \begin{cases} f_{\mathbf{n}_j}^2 - g_{\mathbf{n}_j}^2 \\ f_{\mathbf{n}_j} f_{\mathbf{n}_k} - g_{\mathbf{n}_j} g_{\mathbf{n}_k}, \\ f_{\mathbf{n}_j} f_{\mathbf{n}_k} + g_{\mathbf{n}_j} g_{\mathbf{n}_k} \end{cases} \qquad \mathbf{b}'_s := \begin{cases} f_{\mathbf{n}_j} g_{\mathbf{n}_j} \\ g_{\mathbf{n}_j} f_{\mathbf{n}_k} + f_{\mathbf{n}_j} g_{\mathbf{n}_k}, \\ g_{\mathbf{n}_j} f_{\mathbf{n}_k} - f_{\mathbf{n}_j} g_{\mathbf{n}_k} \end{cases}$$

$$\text{if} \qquad \mathbf{m}_s := \begin{cases} 2\mathbf{n}_j \\ \mathbf{n}_j + \mathbf{n}_k, & j < k \\ \mathbf{n}_j - \mathbf{n}_k, & j < k \end{cases}$$

respectively.

(3) *In particular, the set* $\mathcal{A}_\lambda(T^m)$ *contains the set* $\mathcal{A}_{0_\lambda}(T^m)$ *of all equivalence classes of full eigenmaps defined by*

$$T^m \ni \mathbf{x} \mapsto \frac{1}{\sqrt{p}}(\sqrt{a_1+1}\exp(2\pi i \mathbf{n}_1 \cdot \mathbf{x}), \cdots, \sqrt{a_p+1}\exp(2\pi i \mathbf{n}_p \cdot \mathbf{x})) \in S^{2p-1},$$

where $a_j \in R$ *satisfy* $a_j + 1 \ge 0, \quad 1 \le j \le p$, *and* $\sum_{j=1}^{p} a_j = 0$.

THEOREM D. *The necessary and sufficient conditions for an eigenmap* ϕ *in* (3) *of Thoerem* B

$$T^m \ni \mathbf{x} \mapsto \frac{1}{\sqrt{p}}(\sqrt{a_1+1}\exp(2\pi i \mathbf{n}_1 \cdot \mathbf{x}), \cdots, \sqrt{a_p+1}\exp(2\pi i \mathbf{n}_p \cdot \mathbf{x})) \in S^{2p-1},$$

to be an isometric immersion of (T^m, g_Λ) *into* (S^{2p-1}, can) *are*

$$(1.2) \qquad < \mathbf{a}_k, \mathbf{a}_l >= 4\pi^2 p^{-1} \sum_{j=1}^{p}(a_j+1) n_{k_j} n_{l_j}, \quad 1 \le k, l \le m,$$

where $\mathbf{n}_j = \sum_{i=1}^{m} \xi_i n_{ij}, 1 \leq j \leq p$, *and* $a_j \in R$ *satisfy* $a_j + 1 \geq 0$ $(1 \leq j \leq p)$ *and* $\sum_{j=1}^{p} a_j = 0$.

2. Eigenmaps of flat 3-tori into spheres

2.1 In this section, take the domain to be a flat 3-torus $T^3 = R^3/(Z\mathbf{a}_1 + Z\mathbf{a}_2 + Z\mathbf{a}_3)$, $\mathbf{a}_i = c_i\mathbf{e}_i (1 \leq i \leq 3)$, where $\{\mathbf{e}_i\}_{i=1}^{3}$ is the standard basis. Then we obtain from Theorem C.

THEOREM 2.1. *Let* (T^3, g_Λ) *be a flat 3-torus with* $T^3 = R^3/(Z\mathbf{a}_1 + Z\mathbf{a}_2 + Z\mathbf{a}_3)$, *where* $\mathbf{a}_i = c_i\mathbf{e}_i (1 \leq i \leq 3)$. *Then,*

(1) *if* ϕ *is a full eigenmap of* (T^3, g_Λ) *into* (S^n, can) *with constant energy density* $\frac{\lambda}{2}$, *then* λ *is an eigenvalue of* $\triangle$ *of* (T^3, g_Λ), *and* $n \leq 2p - 1$, *where* $2p = \dim\{f \in C^\infty(T^3) | \triangle f = \lambda f\}$.

(2) *Assume that* $\lambda = 4\pi^2 \| \sum_{j=1}^{3} \xi_j n_{ji} \|^2$, $1 \leq i \leq p$, *where* $\{\xi_1, \xi_2, \xi_3\}$ *is the dual basis of* $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. *Then,* $\mathcal{A}_{0\lambda}(T^3)$ *is exhaused by*

$$
\begin{aligned}
T^3 \ni & x_1 \mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 \\
& \mapsto p^{-1/2}(\sqrt{a_1 + 1}\exp(2\pi i(n_{11}x_1 + n_{21}x_2 + n_{31}x_3)), \\
& \quad \cdots, \sqrt{a_p + 1}\exp(2\pi i(n_{1p}x_1 + n_{2p}x_2 + n_{3p}x_3)) \,) \in S^{2p-1},
\end{aligned}
$$

where $a_j \in R$ *satisfy* $a_j + 1 \geq 0$, $1 \leq j \leq p$ *and* $\sum_{j=1}^{p} a_j = 0$.

Moreover, we have from Theorem 2.1 and Theorem D.

THEOREM 2.2. *Let* (T^3, g_Λ) *be as in Theorem 2.1. Then,*

(1) *if* ϕ *is a full isometric minimal immersion of* (T^3, g_Λ) *into* (S^n, can), *then 3 is an eigenvalue of* $\triangle$ *of* (T^3, g_Λ), *and* $n \leq 2p - 1$, *where* $2p = \dim\{f \in C^\infty(T^3) | \triangle f = 3f\}$.

(2) *The set* $\beta(T^3) \cap \mathcal{A}_{0\lambda}$ *is parametrized by the set of all p-vectors* $(a_1, a_2, \cdots, a_p)$ *in* R^p *with the following conditions:*

(i) $a_j + 1 \geq 0$, $1 \leq j \leq p$, $\sum_{j=1}^{p} a_j = 0$,

(ii)

$$\sum_{j=1}^{p}(a_j+1) < n_{1j}\xi_1, n_{1j}\xi_1 >$$
$$= \sum_{j=1}^{p}(a_j+1) < n_{2j}\xi_2, n_{2j}\xi_2 >$$
$$= \sum_{j=1}^{p}(a_j+1) < n_{3j}\xi_3, n_{3j}\xi_3 >= (p/4\pi^2),$$

(iii)

$$\sum_{j=1}^{p}(a_j+1)n_{1j}n_{2j} = \sum_{j=1}^{p}(a_j+1)n_{1j}n_{3j}$$
$$= \sum_{j=1}^{p}(a_j+1)n_{2j}n_{3j} = 0.$$

The corresponding immersions are the same as in Theorem 2.1

EXAMPLE 2.3. $T^3 = R^3/(Z\mathbf{a}_1 + Z\mathbf{a}_2 + Z\mathbf{a}_3), \mathbf{a}_i = 2\pi\mathbf{e}_i (1 \leq i \leq 3)$. Then $\beta(T^3) \cap \mathcal{A}_{03}$ is exhausted by

$$T^3 \ni x\mathbf{a}_1 + y\mathbf{a}_2 + z\mathbf{a}_3 \mapsto \frac{1}{2}(\exp 2\pi i(x+y+z), \exp 2\pi i(x+y-z),$$
$$\exp 2\pi i(x-y+z), \exp 2\pi i(x-y-z)).$$

2.2 Now, let us focus on the eigenmaps whose images are contained in S^5. Due to Theorem 2.1, we may restrict ourselves to eigenmaps whose images are included in

$$\left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in R^{n+1} | \mathbf{x} \in R^6, \|\mathbf{x}\| = 1 \right\},$$

where $\mathbf{0}$ is the origin of R^{n-5}, if necessary, permuting the coordinates $(x_1, x_2, \cdots, x_{n+1})$ of R^{n+1}. Then we may put $a_4 = a_5 = \cdots = a_p = -1, a_3 = (p-3) - a_1 - a_2$, where the parameter a_1 and a_2 satisfies $p - 2 \geq a_1 + a_2$ and $a_1 \geq -1$, and $a_2 \geq -1$ in Theorem 2.1. Then we obtain :

THEOREM 2.4. *For* $(T^3, g_\Lambda), (T^3 := R^3/(Z\mathbf{a}_1 + Z\mathbf{a}_2 + Z\mathbf{a}_3), \mathbf{a}_i = c_i\mathbf{e}_i(1 \le i \le 3))$, *the set* $\{[\phi] \in \mathcal{A}_{0\Lambda}(T^3) | \phi(T^3) \subset S^5\}$ *is exhausted by the following :*

(2.1)

$$T^3 \ni x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 \mapsto p^{-1/2}\left(\sqrt{a_1+1}\exp\left(2\pi i\sum_{i=1}^{3} x_i n_{i1}\right),\right.$$

$$\left.\sqrt{a_2+1}\exp\left(2\pi i\sum_{i=1}^{3} x_i n_{i2}\right), \sqrt{p-a_1-a_2-2}\exp\left(2\pi i\sum_{i=1}^{3} x_i n_{i3}\right)\right),$$

where the parameters a_1 *and* a_2 *satisfy* $a_1 \ge -1, a_2 \ge -1$ *and* $(p-2) \ge a_1 + a_2$, *and* λ *is the eigenvalue of* Δ *of* (T^3, g_Λ) *with multiplicity, say* $2p$, *and the integers* $n_{ji}(1 \le j, i \le 3)$ *satisty* $\lambda = 4\pi^2\|\sum_{i=1}^{3}\xi_j n_{ji}\|^2, (i = 1, 2, 3)$.

Next consider the injective eigenmaps of a flat 3-torus $T^3 = R^3/\Lambda, (\Lambda = Z\mathbf{a}_1 + Z\mathbf{a}_2 + Z\mathbf{a}_3, a_i = c_i\mathbf{e}_i(1 \le i \le 3)$, into (S^5, can). Then

THEOREM 2.5. *The range-equivalence classes of the injective full eigenmaps of* $(R^3/(Z\mathbf{a}_1 + Z\mathbf{a}_2 + Z\mathbf{a}_3), g_\Lambda), \mathbf{a}_i = c_i\mathbf{e}_i(i = 1, 2, 3)$, *into* (S^5, can) *are exhausted by the following :*

$$T^3 := R^3/\Lambda \ni x\mathbf{a}_1 + y\mathbf{a}_2 + z\mathbf{a}_3 \mapsto p^{-1/2}(\sqrt{a_1+1}\exp(2\pi i x),$$

$$\sqrt{a_2+1}\exp(2\pi i y), \sqrt{p-a_1-a_2-2}\exp(2\pi i z)).$$

Here, the flat torus (T^3, g_Λ) *must be equilateral, i.e.,* $\|\xi_1\| = \|\xi_2\| = \|\xi_3\|$, *where* $\{\xi_1, \xi_2, \xi_3\}$ *is the dual basis of* $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, $2p$ *is the multiplicity of the eigenvalue* $4\pi^2\|\xi_i\|^2, (i = 1, 2, 3)$, *of* Δ, *and the parameters* a_1 *and* a_2 *satisfy* $a_1 > -1, a_2 > -1$, *and* $(p-2) > a_1 + a_2$.

Theorem 2.5 is immediate from the following Lemma:

LEMMA 2.6. *The necessary and sufficient condition for the eigenmaps* π *of the form (2.1) to be injective is*

$$|A| = \pm 1, \qquad \text{where } A = \begin{pmatrix} n_{11} & n_{21} & n_{31} \\ n_{12} & n_{22} & n_{32} \\ n_{13} & n_{23} & n_{33} \end{pmatrix}.$$

For the proof of Lemma 2.6, note that

$$\phi \text{ is injective} \Longleftrightarrow A(\Omega_0 - (0)) \cap Z^3 = \emptyset,$$

where

$$\Omega_0 = \left\{ \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \in R^3 \middle| -1 < d_i < 1 (1 \le i \le 3) \right\}$$

and

$$Z^3 = \left\{ \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \in R^3 \middle| m_i \in Z (1 \le i \le 3) \right\}.$$

Then we get Lemma 2.6 from the following:

SUBLEMMA 2.7.

(1) $|A| = 0 \Longrightarrow A(\Omega_0 - (0)) \cap Z^3 \neq \emptyset$,
(2) $|A| = \pm 1 \Longrightarrow A(\Omega_0 - (0)) \cap Z^3 = \emptyset$,
(3) *Otherwise,* $A(\Omega_0 - (0)) \cap Z^3 \neq \emptyset$.

Proofs of (1) and (2) in Sublemma 2.7 are simple. So we omit it. (3) follows from Minkowski's Convex Body Theorem :

Minkowski's Convex Body Theorem (cf.[4, p.16]). *Let $K \subset R^n$ be a domain which is convex and symmetric about the origin* $\mathbf{0}$. *Assume that $Vol(K) > 2^n Vol(R^n/\Lambda)$, where Λ is a lattice of R^n. Then K contains a non-zero lattice point of Λ.*

Proof of Theorem 2.5. (continued). By Lemma 2.6, we only may consider the eigenmap of the form (2.1) with $|A| = \pm 1$. Take another basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of R^3 as $(\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3) := (\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3)A^{-1}$. Then, $\Lambda = Z\mathbf{a}_1 + Z\mathbf{a}_2 + Z\mathbf{a}_3$. Using $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, the eigenmap can be written as

$$x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 \mapsto p^{-1/2}(\sqrt{a_1+1}\exp(2\pi i x_1),$$
$$\sqrt{a_2+1}\exp(2\pi i x_2), \sqrt{p - a_1 - a_2 - 2}\exp(2\pi i x_3)),$$

and $\exp(2\pi i x_j), (1 \le j \le 3)$, must be the eigenfuctions of $\triangle$ of $R^3/\Lambda, (\Lambda = Z\mathbf{b}_1 + Z\mathbf{b}_2 + Z\mathbf{b}_3)$, which implies the equilaterality of the torus. Thus we obtain Theorem 2.5.

Moreover, we obtain from (2) of Theorem 2.2 and Theorem 2.5 :

COROLLARY 2.8. *The class of injective eigenmaps belonging to $\beta(T^3)$ $\cap \mathcal{A}_{0\lambda}(T^3)$ consists only of the generalized Clifford torus, $c := c_1 = c_2 = c_3 = 2\pi/\sqrt{3}$, and*

$$T^3 = R^3/cZ^3 \ni \mathbf{x} = \sum_{i=1}^{3} x_i \mathbf{a}_i \mapsto \frac{1}{\sqrt{3}}(\exp(2\pi i x_1), \exp(2\pi i x_2), \exp(2\pi i x_3)) \in S^5.$$

References

1. M. P. do Carmo and N. R. Wallach, *Minimal immersions of spheres into spheres,* Ann. Math., **93**(1971), 43–62.
1. J. Eell and L. Lemaire, *A report on harmonic maps,* Bull. London Math. Soc., **10**(1978), 1–68.
3. Joon-Sik Park and H. Urakawa, *Classification of harmonic mappings of constant energy density into spheres,* M.S.R.I. 04208-90 Berkeley, California, April 1990.
4. J. Milnor and D. Husemoller, *Symmetric Bilinear Forms,* Ergebnisse Math., no 73, Springer, 1973.
5. R. T. Smith, *Harmonic mappings of spheres,* Amer. J. Math., **97**(1975), 365–385.
6. T. Takahashi, *Minimal immersions of Riemannian manifolds,* J Math. Soc. Japan **4**(1966), 380–385.
7. G. Tóth and G. D'Ambra, *Parameter space for harmonic maps of constant energy density into spheres.* Geometriae Dedicata, **17**(1984), 61–67.

†DEPARTMENT OF MATHEMATICS, PUSAN UNIVERSITY OF FOREIGN STUDIES, PUSAN 608-060, KOREA

‡DEPARTMENT OF MATHEMATICS, CHUNGBUK NATIONAL UNIVERSITY, CHEONGJU 360-763, KOREA