Bull. Korean Math. Soc. **29** (1992), No. 1, pp. 9–13

CRITICAL METRICS ON NEARLY KAEHLERIAN MANIFOLDS

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The study of the integral of the scalar curvature R, $J(g) := \int_M R dV_g$, as a function on the set of all Riemannian metrics with the same volume on a compact manifold, is now classical and it is well-known ([4]) that the critical points of $J(g)$ are the Einstein metrics; moreover, other functions of the curvature have been taken as integrands and studied in [1]-[6], etc. For example, M. Berger([1]) and Y. Mutō ([5]) investigated the critical point conditions of the functions

$$\int_M R^2 dV_g, \qquad \int_M R_{ji}R^{ji} dV_g, \qquad \int_M R_{kjih}R^{kjhi} dV_g,$$

where R_{ji} and R_{kjih} denote the Ricci and curvature tensors expressed in components, respectively. On the other hand D. E. Blair and S. Ianus ([3]) gave some results about the function $\int_M (R - R^*) dV_g$ on a symplectic manifold, where R^* is the $*$-scalar curvature defined by $R^* := R_{kjih}J^{ji}J^{kh}$ for an almost complex structure J.

In this paper, we consider the function related with almost hermitian structure on a compact complex manifold. More precisely, on a 2n-diminsional complex manifold M admitting 2-form Ω of rank 2n everywhere, assume that M admits a metric g such that $g(JX, JY) = g(X, Y)$, that is, assume that g defines an hermitian structure on M admitting Ω as fundamental 2-form-the 'almost complex structure' J being determined by g and Ω : $g(X,Y) = \Omega(X, JY)$. We consider the function $I(g) := \int_M \|N\|^2 dV_g$, where $\|N\|$ is the norm of Nijenhuis tensor N defined by (J, g). Then, for this function restricted on the set of all nearly Kaehler mitrics on M, we can show the following theorem;

Received December 24, 1990.
This reaearch was partially supported by TGRC-KOSEF.

THEOREM. *Let M be a compact almost hermitian manifold and $\mathcal{M}$ the set of nearly Kaehler metrics, Then a nearly Kaehler metric is a critical point of the function $I(g)$ on $\mathcal{M}$ if and only if it is a Kaehler metric. Moreover, the critical metric is unique.*

1. Let M be a compact almost hermitian manifold with fundamental 2-form Ω and $\dim M = 2n$, i.e.,

$$g(X, JY) = \Omega(X, Y) \text{ and } g(X, Y) = g(JX, JY)$$

where J is an almost complex structure. We remark that g and J are constructed simultaneously by polarization of Ω evaluated on a local orthonormal basis of an arbitrary Riemannian metric. It is well-known that all hermitian metrics have the same volume element so that we may say $\int_M dV_g = 1$. The set $\mathcal{A}$ of all hermitian metrics is infinite dimensional and is totally geodesic in the set of all Riemannian metrics on M. From now on we consider the function

$$I(g) := \int_M \|N\|^2 dV_g \text{ defined on } \mathcal{A}.$$

In the following we use local coordinates and a tensor is expressed in its components with respect to the natural frame. When we take a C^∞ curve $g(t)$ in $\mathcal{A}$, we get several tensor fields defined by

$$D_{ji} := \frac{\partial g_{ji}}{\partial t},\ D_i{}^h := D_{ik}g^{kh},\ D^{ih} := D_k{}^h g^{ki},$$

$$D_{ji}{}^h := \frac{1}{2}(\nabla_j D_i{}^h + \nabla_i D_j{}^h - \nabla^h D_{ji}),\ D_{kji}{}^h := \nabla_k D_{ji}{}^h - \nabla_j D_{ki}{}^h,$$

where ∇ denotes the covariant differentiation with respect to the metric g and $\nabla^h = g^{hi}\nabla_i$. If $g(t)$ is a path in $\mathcal{A}$ with $g(0) = g$, then we have

$$(1.1)\quad \frac{\partial J_{ji}}{\partial t} = 0,\quad \frac{\partial J_j{}^h}{\partial t} = -J_{ji}D^{ih},\quad D_{ba}J_i{}^a D_j{}^b, = -D_{ji},\quad g^{ji}D_{ji} = 0.$$

In general, any symmetric tensor field D satisfying $DJ + JD = 0$ is tangent to some curve in $\mathcal{A}$ (cf. [2]).

Using $\dfrac{\partial\{j^h i\}}{\partial t} = D_{ji}{}^h$ and (1.1), we have

$$\frac{\partial \nabla_j J_i{}^h}{\partial t} = -(\nabla_j J_{it})D^{th} - J_{it}(\nabla_j D^{th}) - D_{ji}{}^t J_t{}^h + D_{jt}{}^h J_i{}^t. \tag{1.2}$$

Hence we have

$$\begin{aligned}
\frac{dI(g)}{dt} &= \int_M \left[\frac{\partial(N_{jk}{}^i N_{bc}{}^a g^{bj} g^{ck} g_{ai})}{\partial t} + \frac{1}{2}\|N_{kj}{}^i\|^2 g^{ba} D_{ba}\right] dV_g \\
&= \int_M N_{bc}{}^a \Big\{ 2\frac{\partial N_{jk}{}^i}{\partial t} g_{ai} g^{bj} g^{ck} - 2N_{jk}{}^i D^{ck} g^{bj} g_{ai} \\
&\qquad + N_{jk}{}^i D_{ai} g^{bj} g^{ck} \Big\} dV_g \\
&= \int_M N_{bc}{}^a \{-2D_{ai} J^{bt} \nabla_t J^{ci} + 2J_{th} D_a{}^h \nabla^b J^{ct} - 4J^{bt} J^{ci} \nabla_t D_{ai} \\
&\qquad - 4J_s{}^b D^{st} \nabla_t J_a{}^c + 4J_{at} D^{st} \nabla^b J_s{}^c + 4J_{at} J_s{}^c \nabla^b D^{st} \\
&\qquad - 2D^{ck} J^{bt} \nabla_t J_{ka} - 2D^{ck} J_k{}^t \nabla_t J_a{}^b - 2D^{ck} J_{at} \nabla^b J_k{}^t \\
&\qquad + 2D^{ck} J_{at} \nabla_k J^{bt}\} dV_g \\
&= \int_M [8D_{ai} J_s^a J^{bt} (\nabla_b J_c{}^s)(\nabla_t J^{ci}) + 8J_b{}^a (\nabla_t D_{ai})(\nabla^i J^{tb}) \\
&\qquad - 8J^{ic} (\nabla_t D_{ai})(\nabla_c J^{ta}) + 4D^{ht} (\nabla_h J_c{}^a)(\nabla_t J_a{}^c) \\
&\qquad - 8D^{ht} (\nabla_t J_c{}^s)(\nabla_s J_h{}^c) + 4D^{ht} J_s{}^a J_h{}^b (\nabla_t J_a{}^c)(\nabla_b J_c{}^s) \\
&\qquad + 8D^{ht} (\nabla^b J_h{}^c)(\nabla_b J_{ct})] dV_g.
\end{aligned} \tag{1.3}$$

2. In this section we will consider only nearly Kaehler structure. By definition a nearly Kaehler structure is an almost hermitian structure (J, g) which satisfies

$$(\nabla_X J)Y + (\nabla_Y J)X = 0$$

for any vector fields X and Y on M.

If (J,g) is a nearly Kaehler structure, then we have

$$J_b{}^a(\nabla_t D_{ai})(\nabla^i J^{tb}) - J^{ic}(\nabla_t D_{ai})(\nabla_c J^{ta}) = 0,$$

and consequently from (1.3)

$$\begin{aligned}\frac{dI(g)}{dt} = \int_M D_{ht}[&8J_s{}^h J^{ba}(\nabla_b J_c{}^s)(\nabla_a J^{ct}) + 4(\nabla^h J_c{}^a)(\nabla^t J_a{}^c)\\ &- 8(\nabla^t J^{sc})(\nabla_s J_c{}^h) + 4J_s{}^a J^{hb}(\nabla^t J_a{}^c)(\nabla_b J_c{}^s)\\ &+ 8(\nabla^b J^{hc})(\nabla_b J_c{}^t)]dV_g\end{aligned}$$

$$= -32\int_M D_{ht}(\nabla^s J^{ct})(\nabla_s J_c{}^h)dV_g. \tag{2.1}$$

On the other hand D_{ji} must satisfy the only condition

$$\int_M D_{ji}g^{ji}dV_g = 0. \tag{2.2}$$

Let $\bar{g}$ be a nearly Kaehler metric such that, for every C^∞ curve $g(t)$ of M satisfying $g(0) = \bar{g}$, $I(g(t))$ has vanishing derivative at $t = 0$. Then from (2.1) and (2.2) it follows that

$$-32(\nabla^s J^{ct})(\nabla_s J_c{}^h) = cg^{ht},\ c = const. \tag{2.3}$$

for this metric $\bar{g}$ and consequently

$$2nc = -32(\nabla^s J^{ct})(\nabla_s J_{ct}).$$

By the way, transvecting $J_h{}^e J_{et}$ to (2.3) and using the fact that $(J,\bar{g})$ is nearly Kaehlerian, we can obtain

$$c = -32(\nabla^s J^{ct})(\nabla_s J_{ct})$$

and so $c = 0$, that is, $\nabla_s J_{ct} = 0$ for this metric $\bar{g}$. Hence we can assert that the formal part of main theorem stated in introduction is true.

Next, we differentiate again (2.1) with respect to the parameter t. Taking account of (J, g) being nearly Kaehlerian, we can easily find

$$\begin{aligned}\frac{d^2 I(g)}{dt^2} = -32 \int_M \Big[& \frac{\partial D_{hi}}{\partial t}(\nabla^s J^{ci})(\nabla_s J_c{}^h) + D_{ht}\{-D^{si}(\nabla_i J^{ct})(\nabla_s J_c{}^h) \\ & + D_i^{sc} J^{it}(\nabla_s J_c{}^h) + D_i^{st} J^{ci}(\nabla_s J_c{}^h) \\ & + (\nabla^s J^{ct})(\nabla_s D_{cb}) J^{bh} + (\nabla^s J^{ct})(\nabla_s J^{bh}) D_{cb} \\ & - D_{sc}{}^b J_b{}^h(\nabla^s J^{ct}) + D_{sb}{}^h J_c{}^b(\nabla^s J^{ct})\}\Big] dV_g \\ = 32 \int_M & D_{ht} D_{kc}(\nabla_s J^{hk})(\nabla^s J^{tc}) dV_g.\end{aligned}$$

If we put $D_{ji} = f g_{ji}$, where f is a C^∞ function on M such that $\int_M f dV_g = 0$, then $\dfrac{d^2 I(g)}{dt^2} > 0$. This means that the last main theorem is also true.

References

1. M. Berger, *Quelques formules de variation pour une structure riemannienne,* Ann. Sci. École Norm. Sup., (4) **3**(1970), 285–294.
2. D. E. Blair, *On the set of metrics associated to a symplectic or contact form,* Bull. Inst. Math. Acad. Sinica, **11**(1983), 297–308.
3. D. E. Blair and S. Ianus, *Critical associated metrics on symplectic manifolds,* Contemporary Math. **51**(1986), 23–29.
4. Y. Mutō, *On Einstein metrics,* J. Diff. Geo., **9**(1974), 521–530.
5. Y. Mutō, *Curvature and critical Riemannian metrics,* J. Math. Soc. Japan, **26**(1974), 686–697.
6. Y. Mutō, *Riemannian submersions and critical Riemannian merics,* J. Math. Soc. Japan, **29**(1977), 493–511.

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