Bull. Korean Math. Soc. **29** (1992), No. 1, pp. 1–7

THE ASYMPTOTIC BEHAVIOR OF NON–LINEAR DISSIPATIVE HYPERBOLIC EQUATIONS

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1. Introduction

Let $\mathbb{R}$ be the set of all real numbers, $t_0 > 0$ in $\mathbb{R}$ and $\Omega = [t_{0,\infty}[\times[0,\pi]$. Let $C_B^2(\Omega)$ be the space of all functions $u : \Omega \longrightarrow \mathbb{R}$ which have bounded continuous partial derivatives up to order 2 with respect to both variables on Ω.

In this note, we will investigate the asymptotic stability of global solutions of non-linear dissipative hyperbolic equations of the form

$$\beta u_t + u_{tt} - u_{xx} + g(u) = 0 \text{ in } \Omega \tag{1.1}$$

where $\beta(>0) \in \mathbb{R}, u = u(t,x)$ and $g : \Omega \longrightarrow R$ is continuous.

A *global solution* of the problem on Ω for (1.1) will be $u \in C_B^2(\Omega)$ such that u satisfies (1.1) on Ω and satisfies Dirichlet boundary condition

$$u(t,0) = u(t,\pi) = 0, \quad t > t_0.$$

In [7], the author established the existence of a weak solution to the periodic-Dirichlet boundary value problem for the equations of the form

$$\beta u_t + u_{tt} - u_{xx} + g(u) = h(t,x)$$

with a generalized sign condition and superlinear growth in g. Our main result is related to the above problem. For stability result, we assume $h(t,x) = 0$ and it is also another interesting question whether we could establish the asymptotic stability for the equations with non-zero forcing term $h(t,x)$ with appropriate conditions.

Received December 12, 1990.

This research was supported by a grant from KOSEF, 1990.

There have been a few authors dealing the asymptotic behavior of this type of hyperbolic equations, for example, Ficken and Fleishman [2], Haraux [3] and Rabinowitz [8]. In particular, our work is related to Rabinowitz's work in [8]. Rabinowitz treated the asymptotic stability of global solutions for the equaions of the form (1.1) with $g = \varepsilon F(t, x, u)$. He discussed the asymptotic stability of any global solutions satisfying Dirichlet condition in x but his result is valied for sufficiently small ε. In our main result, our non-linear term is independent of such a parameter and we will prove that any global solution satisfying Dirichlet condition in x to the above equation (1.1) converges to zero exponentially. We assume only

$$(H_1) \qquad ug(u) \geq 0 \quad \text{for all } u \in \mathbb{R} \quad \text{and}$$

$$(H_2) \qquad G(u) \leq ug(u) \quad \text{for all } u \in \mathbb{R}$$

where $G(u) = \int_0^u g(s)\,ds$.
The assumption (H_1) is nothing but a sign condition and the assumption (H_2) contains both sublinear and superlinear growth conditions in g and those two conditions (H_1) and (H_2) play important parts in driving an energy inequality. For this result, we will assume some more regularity of solutions than we do for our existence result in [7].

Our proof is based on the so called "energy method" and the main key in our proof is to set up an energy function as a function of t and an appropriate parameter ε and driving a simple differential inequality by using Dirichlet condition in x.

2. Main Result

THEOREM. *Assume g satisfies (H_1) and (H_2) and also assume $g(0) = 0$. If $v \in C_B^2(\Omega)$ is any global solution to (1.1), then $v(t,x) \to 0$ uniformly as $t \to +\infty$ and in this case $v(t,x)$ decays exponentially.*

Proof. Since $g(0) = 0, u \equiv 0$ is a solution of (1.1). Let $v(t,x)$ be any global solution and consider an energy function defined by

$$w(t) = \frac{1}{2}\int_0^\pi [v_t^2 + v_x^2 + \varepsilon\beta v^2 + 2\varepsilon v v_t]\,dx + \int_0^\pi G(v)\,dx$$

where $\varepsilon > 0$.
Differentiate $w(t)$ with respect to t, then, we have,

$$\frac{d}{dt}w(t) = \int_0^\pi [v_t v_{tt} + v_x v_{tx} + \varepsilon\beta v v_t + \varepsilon v_t^2 + \varepsilon v v_{tt} + g(v)v_t]\, dx$$

Since $v_t(t,0) = v_t(t,\pi) = 0$, we have,

$$\int_0^\pi v_x v_{tx}\, dx = -\int_0^\pi v_t v_{xx}\, dx\,.$$

Therefore

$$\frac{d}{dt}w(t) = \int_0^\pi v_t[v_{tt} - v_{xx} + g(v)]\, dx + \varepsilon\int_0^\pi v_t^2\, dx + \int_0^\pi \varepsilon v[\beta v_t + v_{tt}]\, dx.$$

Since $v(t,x)$ satisfies equation (1.1),

$$\frac{d}{dt}w(t) = -(\beta - \varepsilon)\int_0^\pi v_t^2\, dx + \varepsilon\int_0^\pi v v_{xx}\, dx - \varepsilon\int_0^\pi v g(v)\, dx.$$

Using the Dirichlet boundary condition, we get

$$\varepsilon\int_0^\pi v v_{xx}\, dx = -\varepsilon\int_0^\pi v_x^2\, dx.$$

Hence

$$\begin{aligned}\frac{d}{dt}w(t) &= -(\beta - \varepsilon)\int_0^\pi v_t^2\, dx - \varepsilon\int_0^\pi v_x^2\, dx - \varepsilon\int_0^\pi v g(v)\, dx \\ &= -\left[(\beta - \varepsilon)\int_0^\pi v_t^2\, dx + \varepsilon\int_0^\pi v_x^2\, dx + \varepsilon\int_0^\pi v g(v)\, dx\right].\end{aligned}$$

Since $ug(u) \geq 0$ for all u,

$$\frac{d}{dt}w(t) \leq 0 \quad \text{for all } 0 \leq \varepsilon \leq \beta.$$

Furthermore

$$
\begin{aligned}
-\frac{d}{dt}w(t) &= (\beta-\varepsilon)\int_0^\pi v_t^2\,dx + \varepsilon\int_0^\pi v_x^2\,dx + \varepsilon\int_0^\pi vg(v)\,dx \\
&\geq \min\{2(\beta-\varepsilon),\varepsilon\}\left[\int_0^\pi \frac{v_t^2}{2}\,dx + \int_0^\pi \frac{v_x^2}{2}\,dx + \int_0^\pi vg(v)\,dx\right] \\
&\geq \min\{2(\beta-\varepsilon),\varepsilon\}\left[\int_0^\pi \frac{v_t^2}{2}\,dx + \int_0^\pi \frac{v_x^2}{2}\,dx + \int_0^\pi G(v)\,dx\right]
\end{aligned}
$$

since $G(u) \leq ug(u)$ for all $u \in R$.
Again, by the boundary condition, we have

$$v(t,x) = \int_0^\pi v_x(t,s)\,ds.$$

Hence

$$
\begin{aligned}
|v(t,x)| &\leq \int_0^x |v_x(t,s)|\,ds \\
&\leq \int_0^\pi |v_x(t,x)|\,dx \\
&\leq \pi^{1/2}\left[\int_0^\pi |v_x(t,x)|^2\,dx\right]^{1/2}.
\end{aligned}
$$

Thus, we have

$$|v(t,x)|^2 \leq \pi\int_0^\pi |v_x(t,x)|^2\,dx.$$

Therefore, we obtain

$$
\begin{aligned}
&\int_0^\pi |v(t,x)|^2\,dx \leq \pi^2\int_0^\pi |v_x(t,x)|^2\,dx, \text{ or} \\
&\int_0^\pi v^2\,dx \leq \pi^2\int_0^\pi v_x^2\,dx.
\end{aligned}
$$

Now

$$\begin{aligned}
w(t) &= \frac{1}{2}\int_0^\pi [v_t^2 + v_x^2 + \varepsilon\beta v^2 + 2\varepsilon v v_t]\,dx + \int_0^\pi G(v)\,dv \\
&\le \frac{1}{2}\int_0^\pi [v_t^2 + v_x^2 + \varepsilon\beta\pi^2 v_x^2 + \varepsilon v^2 + \varepsilon v_t^2]\,dx + \int_0^\pi G(v)\,dx \\
&\le \frac{1}{2}\int_0^\pi [v_t^2 + v_x^2 + \varepsilon\beta\pi^2 v_x^2 + \varepsilon\pi^2 v_x^2 + \varepsilon v_t^2]\,dx + \int_0^\pi G(v)\,dx \\
&= (1+\varepsilon)\int_0^\pi \frac{v_t^2}{2}\,dx + (1+\varepsilon\pi^2+\varepsilon\beta\pi^2)\int_0^\pi \frac{v_x^2}{2}\,dx + \int_0^\pi G(v)\,dx \\
&= (1+\varepsilon\pi^2+\varepsilon\beta\pi^2)\left[\int_0^\pi \frac{v_t^2}{2}\,dx + \int_0^\pi \frac{v_x^2}{2}\,dx + \int_0^\pi G(v)\,dx\right].
\end{aligned}$$

Hence

$$\int_0^\pi \frac{v_t^2}{2}\,dx + \int_0^\pi \frac{v_x^2}{2}\,dx + \int_0^\pi G(v)\,dx \ge \frac{w(t)}{1+\varepsilon\pi^2+\varepsilon\beta\pi^2}$$

Therefore

$$-\frac{d}{dt}w(t) \ge \frac{\min\{2(\beta-\varepsilon),\varepsilon\}}{1+\varepsilon\pi^2+\varepsilon\beta\pi^2}w(t).$$

Take $0<\varepsilon<\beta$ such that $\varepsilon_0 = \dfrac{\min\{2(\beta-\varepsilon),\varepsilon\}}{1+\varepsilon\pi^2+\varepsilon\beta\pi^2} > 0$, then $-\dfrac{d}{dt}w(t) \ge \varepsilon_0 w(t)$ or

$$\frac{d}{dt}w(t) \le -\varepsilon_0 w(t). \tag{1.3}$$

Multiply $e^{\varepsilon_0 t}$ on the both side of (1.3), to get

$$\frac{d}{dt}w(t)e^{\varepsilon_0 t} + \varepsilon_0 w(t)e^{\varepsilon_0 t} \le 0.$$

Hence $\dfrac{d}{dt}[w(t)e^{\varepsilon_0 t}] \le 0$ and thus $w(t)e^{\varepsilon_0 t}$ is monotone decreasing with respect to t.

On the other hand, since $G(u) = \int_0^u g(s)\,ds \geq 0$, we have

$$
\begin{aligned}
w(t) &= \frac{1}{2}\int_0^\pi [v_t^2 + v_x^2 + \varepsilon\beta v^2 + 2\varepsilon v v_t]\,dx + \int_0^\pi G(v)\,dx \\
&= \frac{1}{2}\int_0^\pi [(v_t + \varepsilon v)^2 + (\varepsilon\beta - \varepsilon^2)v^2 + v_x^2]\,dx + \int_0^\pi G(v)\,dx \\
&> 0.
\end{aligned}
$$

Therefore, $0 \leq w(t)e^{\varepsilon_0 t} \leq w(t_0)e^{\varepsilon_0 t}$, or $w(t) \leq w(t_0)e^{\varepsilon_0(t_0 - t)}$

Now, since $|v(t,x)| \leq \pi^{1/2}\left[\int_0^\pi |v_x(t,x)|^2\,dx\right]^{1/2}$,

$$
\sup_{0 \leq x \leq \pi} |v(t,x)| \leq \pi^{1/2}\left[\int_0^\pi |v_x(t,x)|^2\,dx\right]^{1/2}.
$$

Since $w(t) \geq \frac{1}{2}\int_0^\pi v_x^2\,dx$,

$$
\begin{aligned}
\sup_{0 \leq x \leq \pi} |v(t,x)| &\leq \pi^{1/2}[2w(t)]^{1/2} \\
&\leq \pi^{1/2}[2w(t_0)e^{\varepsilon_0(t_0 - t)}]^{1/2}
\end{aligned}
$$

Hence, as $t \to +\infty$, $\sup_{0 \leq x \leq \pi} |v(t,x)| \to 0$.

This completes our proof.

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