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FUZZY AND LEVEL SUBALGEBRAS OF BCK(BCI)–ALGEBRAS

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The concept of fuzzy sets was introduced by Zadeh in [6]. Das [1] and Rosenfeld [5] applied it to the fundamental theory of groups. Ougen [4] applied the concept of fuzzy set to *BCK*-algebra, and he got some results. In this paper, we study fuzzy subalgebras of a *BCK(BCI)*-algebra. We also define level subalgebra and study their properties.

DEFINITION 1. A *BCI-algebra* is a non-empty set X with a binary operation $*$ and a constant 0 satisfying the axioms:

(1) $\{(x*y)*(x*z)\}*(z*y)=0$,
(2) $\{x*(x*y)\}*y=0$,
(3) $x*x=0$,
(4) $x*y=0$ and $y*x=0$ imply that $x=y$,
(5) $x*0=0$ implies that $x=0$,

for all $x,y,z\in X$. If (5) is replaced by $0*x=0$, then the algebra is called a *BCK-algebra*.

A non-empty subset A of a *BCK(BCI)*-algebra X is called a *subalgebra* of X if $x,y\in A$ implies $x*y\in A$.

DEFINITION 2. ([1]). Let X be a set. A *fuzzy set* in X is a function $\mu : X\to[0,1]$.

DEFINITION 3. ([4], [5]). Let X be a *BCK(BCI)*-algebra. A fuzzy set μ in X is called a *fuzzy subalgebra* of X if, for all $x,y\in X$,

$$\mu(x*y)\geq\min(\mu(x),\mu(y)).$$

LEMMA 4. ([4]). *Let X be a BCK(BCI)-algebra. If $\mu : X\to[0,1]$ is a fuzzy subalgebra of X, then $\mu(x)\leq\mu(0)$ for all $x\in X$.*

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THEOREM 5. *Let μ be a fuzzy set in a BCK(BCI)-algebra X. If μ is a fuzzy subalgebra of X, then the set*

$$A := \{x \in X : \mu(x) = \mu(0)\}$$

is a subalgebra of X.

Proof. Let $x, y \in A$. Then $\mu(x) = \mu(0) = \mu(y)$. Since μ is a fuzzy subalgebra, it follows that

$$\mu(x * y) \geq \min(\mu(x), \mu(y)) = \mu(0).$$

On the other hand, by Lemma 4, we have $\mu(x * y) \leq \mu(0)$. Hence $\mu(x * y) = \mu(0)$, and $x * y \in A$.

THEOREM 6. *The intersection of any set of fuzzy subalgebras of a BCK(BCI)-algebra is a fuzzy subalgebra.*

Proof. Let $\{\mu_i\}$ be a family of fuzzy subalgebras of a *BCK(BCI)*-algebra X. Then, for any $x, y \in X$,

$$\begin{aligned}(\cap\mu_i)(x * y) &= \inf(\mu_i(x * y)) \\ &\geq \inf(\min(\mu_i(x), \mu_i(y))) \\ &= \min(\inf \mu_i(x), \inf \mu_i(y)) \\ &= \min((\cap\mu_i)(x), (\cap\mu_i)(y)),\end{aligned}$$

which completes the proof.

REMARK 7. The union of any set of fuzzy subalgebras need not be a fuzzy subalgebra. Indeed, if μ and ν are fuzzy subalgebras of *BCK(BCI)*-algebra X such that $\mu(x) > \mu(y) > \nu(x) > \nu(y)$ for all $x, y \in X$, then the inequality

$$(\mu \cup \nu)(x * y) \geq \min((\mu \cup \nu)(x), (\mu \cup \nu)(y))$$

does not hold.

THEOREM 8. *Let X be a BCK(BCI)-algebra, A a subset of X, and let μ be a fuzzy set in X such that μ is into $\{0,1\}$, so that μ is the characteristic function of A. Then μ is a fuzzy subalgebra of X if and only if A is a subalgebra of X.*

Proof. Assume that μ is a fuzzy subalgebra of X. Since μ is the characteristic function of A, therefore $\mu(x) = 1 = \mu(y)$ for all $x, y \in A$. Then

$$\mu(x * y) \geq \min(\mu(x), \mu(y)) = 1,$$

and hence $\mu(x * y) = 1$. This implies that $x * y \in A$, and that A is a subalgebra of X. Conversely, suppose that A is a subalgebra of X. Let $x, y \in X$. If $x, y \in A$, then $\mu(x) = 1 = \mu(y)$ and $x * y \in A$. Thus $\mu(x * y) = 1 = \min(\mu(x), \mu(y))$. If $x, y \in X - A$, then $\mu(x) = 0 = \mu(y)$. Thus $\mu(x * y) \geq 0 = \min(\mu(x), \mu(y))$. If $x \in A$ and $y \in X - A$, then $\mu(x) = 1$ and $\mu(y) = 0$. Hence $\mu(x * y) \geq 0 = \min(\mu(x), \mu(y))$. Similarly, for $x \in X - A$ and $y \in A$, we have $\mu(x * y) \geq \min(\mu(x), \mu(y))$. This completes the proof.

DEFINITION 9. Let X and X' be *BCK(BCI)*-algebras. A mapping $f : X \to X'$ is called a *homomorphism* if, for any $x, y \in X$,

$$f(x * y) = f(x) * f(y).$$

DEFINITION 10. ([5]). Let f be a mapping defined on a set X. If μ is a fuzzy set in X, then the fuzzy set ν in $f(X)$ defined by

$$\nu(y) = \sup_{x \in f^{-1}(y)} \mu(x)$$

for all $y \in f(X)$ is called the *image* of μ under f. Similarly, if ν is a fuzzy set in $f(X)$, then the fuzzy set $\mu = \nu \circ f$ in X (i.e., the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in X$) is called the *preimage* of ν under f.

THEOREM 11. *An onto homomorphic preimage of a fuzzy subalgebra is a fuzzy subalgebra.*

Proof. Let $f : X \to X'$ be an onto homomorphism of *BCK(BCI)*-algebras, ν a fuzzy subalgebra of X', and μ the preimage of ν under f.

Then

$$\begin{aligned}\mu(x*y) &= \nu(f(x*y)) \\ &= \nu(f(x)*f(y)) \\ &\geq \min(\nu(f(x)),\nu(f(y))) \\ &= \min(\mu(x),\mu(y))\end{aligned}$$

for all $x, y \in X$. Hence μ is a fuzzy subalgebra of X.

DEFINITION 12. ([5]). A fuzzy set μ in X has *sup property* if, for any subset $T \subset X$, there exists $t_0 \in T$ such that

$$\mu(t_0) = \sup_{t\in T} \mu(t).$$

THEOREM 13. *An onto homomorphic image of a fuzzy subalgebra with sup property is a fuzzy subalgebra.*

Proof. Let $f : X \to X'$ be an onto homomorphism of *BCK(BCI)*-algebras, μ a fuzzy subalgebra of X with sup property, and ν the image of μ under f. Given $x', y' \in X'$, let $x_0 \in f^{-1}(x')$, $y_0 \in f^{-1}(y')$ be such that

$$\mu(x_0) = \sup_{t\in f^{-1}(x')} \mu(t), \quad \mu(y_0) = \sup_{t\in f^{-1}(y')} \mu(t),$$

respectively. Then

$$\begin{aligned}\nu(x'*y') &= \sup_{z\in f^{-1}(x'*y')} \mu(z) \\ &\geq \min(\mu(x_0),\mu(y_0)) \\ &= \min(\sup_{t\in f^{-1}(x')} \mu(t), \sup_{t\in f^{-1}(y')} \mu(t)) \\ &= \min(\nu(x'),\nu(y')).\end{aligned}$$

Hence ν is a fuzzy subalgebra of X'.

DEFINITION 14 ([1]). Let μ be a fuzzy set in a set X. For $t \in [0,1]$, the set

$$\mu_t := \{x \in X : \mu(x) \geq t\}$$

is called a *level subset* of μ.

Note that μ_t is a subset of X in the ordinary sense. The terminology "level set" was introduced by Zadeh.

PROPOSITION 15 ([4]). *Let X be a BCK(BCI)-algebra and μ a fuzzy subalgebra of X. Then the level subset μ_t, for $t \in [0,1]$, $t \leq \mu(0)$, is a subalgebra of X, where "0" in $\mu(0)$ is the constant of X.*

PROPOSITION 16 ([4]). *Let X be a BCK(BCI)-algebra and let μ be a fuzzy set in X such that μ_t is a subalgebra of X for all $t \in [0,1]$, $t \leq \mu(0)$. Then μ is a fuzzy subalgebra of X.*

DEFINITION 17. Let X be a *BCK(BCI)*-algebra and let μ be a fuzzy subalgebra of X. The subalgebras μ_t, $t \in [0,1]$ and $t \leq \mu(0)$, are called *level subalgebras* of μ.

Note that if X is a finite *BCK(BCI)*-algebra, then the number of subalgebras of X is finite whereas the number of level subalgebras of a fuzzy subalgebra μ appears to be infinite. But, since every level subalgebra is indeed a subalgebra of X, not all these level subalgebras are distinct. The next theorem characterises this aspect.

THEOREM 18. *Let μ be a fuzzy subalgebra of a BCK(BCI)-algebra X. Two level subalgebras μ_{t_1}, μ_{t_2} (with $t_1 < t_2$) of μ are equal if and only if there is no $x \in X$ such that $t_1 < \mu(x) < t_2$.*

Proof. Assume that $\mu_{t_1} = \mu_{t_2}$ for $t_1 < t_2$ and that there exists $x \in X$ such that $t_1 < \mu(x) < t_2$. Then $\mu_{t_2} \subset \mu_{t_1}$ and $\mu_{t_1} \neq \mu_{t_2}$, which contradicts the hypothesis. Conversely suppose that there is no $x \in X$ such that $t_1 < \mu(x) < t_2$. Since $t_1 < t_2$, we have $\mu_{t_2} \subset \mu_{t_1}$. Let $x \in \mu_{t_1}$. Then $\mu(x) \geq t_1$, and hence $\mu(x) \geq t_2$, because $\mu(x)$ does not lie between t_1 and t_2. Hence $x \in \mu_{t_2}$, which implies that $\mu_{t_1} \subset \mu_{t_2}$. This completes the proof.

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