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UPPER AND LOWER BOUNDS FOR THE HAUSDORFF MEASURE IN RELATION TO THE PACKING MEASURE

HAN SOO KIM AND YOUNG MI KIM

1. Introduction

The size of sets of zero Lebesgue measure in $\mathbf{R}^n$ can be investigated by several distinct outer measures. The first of these to be extensively developed was Hausdorff measure and the second, namely Packing measure, was established by S. J. Taylor and C. Tricot in 1985.

Packing measure has been used by comparison with Hausdorff measure to study the regularity and rectifiability of sets. In [4], [5], they investigated the relationship between a general measure and its density behaviour. The density behaviour is very useful to know the geometric properties of a fractal set.

In this paper, we consider the case when the Hausdorff measure is not equal to the Packing measure and find a lower bound and an upper bound of Hausdorff measure with respect to the Packing measure.

The first part of this paper is generalization of inequalities in [6] and the second part of this paper considers the lower Hausdorff density of the generalized symmetric Cantor set.

2. Preliminaries

Let the function $h : [0, \infty) \to \mathbf{R}^+$ be continuous, increasing, $h(0) = 0$, and satisfies a smoothness condition : There exists $c_0 > 0$ such that $h(2x) < c_0 h(x)$ for all $0 < x < \frac{1}{2}$.

In this paper, $B(x,r)$ is the ball with center x and radius r in $\mathbf{R}^n$, E_n, E are subsets of $\mathbf{R}^n$, and $|E|$ denotes the diameter of a set E.

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DEFINITION 1. (Hausdorff measure)

$$\mu_\delta^h(E) = \inf\{\sum_n h(|E_n|) : E \subset \cup_n E_n,\ |E_n| \le \delta\}$$

$$\mu^h(E) = \lim_{\delta\to 0} \mu_\delta^h(E).$$

DEFINITION 2. The premeasure of a bounded set E is defined as

$$P^h(E) = \lim_{\delta\to 0}[\sup\{\sum_i h(|B(x_i, r_i)|) : x_i \in E, B(x_i, r_i) \text{ are pairwise disjoint}, 2r_i \le \delta\}].$$

P^h is not outer measure, so by Method I of Munroe, the Packing measure is defined by

$$p^h(E) = \inf\{\sum_n P^h(E_n) : E_n \text{ are bounded },\quad E \subset \cup_n E_n\}.$$

We note that $\mu^h(E) \le p^h(E)$ for a set E.

DEFINITION 3. Lower Hausdorff density function is defined by

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\mu^h(E\cap B(x,r))}{h(2r)}$$

and upper Packing density function is defined by

$$\overline{d}_p(x) = \limsup_{r\to 0} \frac{p^h(E\cap B(x,r))}{h(2r)}$$

3. Results

An upper bound for Hausdorff measure is given in the first theorem.

THEOREM 1. *If* $\{E_n\}$ *are disjoint,* μ^h*-measurable sets such that* $E = \cup_n E_n$*, and if* $\overline{d}_p(x) > a_n > 1$ *for* $x \in E_n$*, a.e.* μ^h*, then* $\mu^h(E) \leq \sum_n a_n^{-1} p^h(E_n)$.

Proof. Since $\overline{d}_p(x) > a_n > 1$ for $x \in E_n$, a.e. μ^h, for each $\delta > 0$, there exist infinitely many $r < \frac{\delta}{2}$ such that $p^h(E_n \cap B(x,r)) > a_n h(2r)$. Let

$$\mathcal{V} = \{B(x,r) : a_n^{-1} p^h(E_n \cap B(x,r)) > h(2r), \quad x \in E_n\}.$$

Then $\mathcal{V}$ is a covering of E_n in the Vitali sense. Using the Vitali covering theorem for Hausdorff measure, $\mu^h(E_n) \leq a_n^{-1} p^h(E_n)$. Since E_n are μ^h-measurable, $\mu^h(E) \leq \sum_n a_n^{-1} p^h(E_n)$.

COROLLARY 2. $\overline{d}_p(x) > a > 1$*, a.e.* μ^h *on* E*, then* $\mu^h(E) \leq a^{-1} p^h(E)$.

By the following theorem, we obtain a lower bound for the Hausdorff measure.

THEOREM 3. *Let* $\{E_n\}$ *be pairwise disjoint,* μ^h*-measurable sets such that* $E = \cup_n E_n$*. If* $\overline{d}_p(x) < a_n$ *for* $x \in E_n$*, a.e.* p^h *and* $\sum_n a_n^{-1} p^h(E_n) < \infty$*, then* $c_0^{-1} \sum_n a_n^{-1} p^h(E_n) \leq \mu^h(E)$*, where* c_0 *such that* $h(2x) < c_0 h(x)$ *for all* $0 < x < \frac{1}{2}$.

Proof. For each $\delta > 0$, let

$$E_{n,\delta} = \{x \in E_n : p^h(E_n \cap B(x,r)) < (a_n - \epsilon)h(2r) \text{ for all } 0 < r \leq \delta \text{ for some } \epsilon > 0\}.$$

Let $\{U_i\}$ be a δ-cover of E_n and thus of $E_{n,\delta}$, the ball B with center x and radius $|U_i|$ certainly contains U_i. Thus

$$p^h(E_n \cap U_i) \leq p^h(E_n \cap B) \leq a_n h(2|U_i|) \leq a_n c_0 h(|U_i|)$$

so that

$$p^h(E_n, \delta) \leq \sum_i \{p^h(E_n \cap U_i) : U_i \text{ intersects } E_{n,\delta}\} \leq c_0 a_n \mu_\delta^h(E_n).$$

But $E_{n,\delta}$ increases to E_n as δ decreases to 0. Hence $c_0^{-1} a_n^{-1} p^h(E_n) \leq \mu^h(E_n)$. Therefore $c_0^{-1} \sum_n a_n^{-1} p^h(E_n) \leq \mu^h(E)$.

In theorem 3, if $h(x) = x^s$, then c_0 becomes 2^s.

COROLLARY 4. *If $\overline{d}_p(x) < a$ for $x \in E$, a.e. p^h and $a^{-1}p^h(E) < \infty$, then $c_0^{-1}a^{-1}p^h(E) \leq \mu^h(E)$.*

THEOREM 5. *Let $E_n \uparrow E, E_n$ be bounded and $0 < \mu^h(E) < \infty$. If $\lim_{n\to\infty} \int_{E_n} \frac{1}{\underline{d}_\mu(x)} d\mu^h(x) = \infty$, then $p^h(E) = \infty$.*

Proof. Let M be a positive integer. There exists a natural number N such that $\int_{E_n} \frac{1}{\underline{d}_\mu(x)} d\mu^h(x) > M$ for all $n \geq N$. Then there exists a simple function $s(x) = \sum_{i=1}^k a_i \chi_{A_i}(x)$ such that $\frac{1}{\underline{d}_\mu(x)} > s(x)$ for all $x \in A_i$ and $\sum_{i=1}^k \mu^h(A_i) > M$, where A_i are disjoint and $a_i = s(x)$ if $x \in A_i$.

Given $\epsilon > 0$, there exists a closed set $F_i \subset A_i$ such that $\mu^h(F_i) > \mu^h(A_i)$-$\frac{\epsilon}{a_i 2^i}$, $i = 1, 2, \cdots, k$. Let r_0 be real number such that

$$r_0 < \frac{1}{3}\text{dist}(F_i, \cup_{m=1, m\neq i}^k F_m).$$

For any $r < r_0$, put

$$\mathcal{V}_i = \{B(x,r) : |2r| < r_0, x \in F_i, h(2r) > a_i \mu^h(A_i \cap B(x,r))\}.$$

For each i, by the Vitali covering theorem, there exist disjoint $\{B(x_j, r_j)\}_{j=1}^{N_i}$ such that

$$\mu^h(F_i \cap (\cup_{j=1}^{N_i} B(x_j, r_j))) > \mu^h(F_i) - \frac{\epsilon}{a_i 2^i}.$$

Then

$$\begin{aligned}
\sum_{i=1}^k \sum_{j=1}^{N_i} h(2r_j) &\geq \sum_{i=1}^k a_i \sum_{j=1}^{N_i} (A_i \cap (B(x_j, r_j))) \\
&\geq \sum_{i=1}^k (\mu^h(F_i) - \frac{\epsilon}{a_i 2^i}) \\
&> \sum_{i=1}^k a_i \mu^h(A_i) - 2\epsilon > M - 2\epsilon.
\end{aligned}$$

So $P^h(E_n) \geq M$ for all $n \geq N$. Therefore $p^h(E) = \infty$.

DEFINITION 6. (Generalized Symmetric Cantor set) Let $[0,1] = E_0 \supset E_1 \supset \cdots$ be a decreasing sequence of sets which have the following properties : for each closed interval I_k of length a_k of E_k, the intervals $I^1_{k+1}, \cdots, I^m_{k+1}$ $(m \geq 2)$ contained in I_k are of equal length a_{k+1} and equally spaced by b_{k+1}. The generalized Symmetric Cantor set E is defined as $E = \cap_n [\cup_{i=1}^{m^n} I^i_n]$,where $\{I^i_n\}$ are the closed intervals of length a_n of E_n.

REMARK 7. Let E be a generalized Symmetric Cantor set. Then if the Hausdorff dimension of E is equal to the Packing dimension, then $s = \lim_{n\to\infty} \dfrac{\log m^{-n}}{\log a_n}$.

REMARK 8. Let E be a generalized Symmetric Cantor set. Then $s = \lim_{n\to\infty} \dfrac{\log m^{-n}}{\log a_n}$ if and only if $\lim_{n\to\infty} \sqrt[n]{a_n} = (\frac{1}{m})^{\frac{1}{s}}$.

Since

$$\underline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \lim_{n\to\infty} \sqrt[n]{a_n},$$

it follows that $\underline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} \leq m^{-\frac{1}{s}}$.

From now on, $h(x) = x^s$, $0 < s < 1$, $\mu^h = \mu^s$, $\underline{d}_\mu = \underline{d}^s_\mu$, $\overline{d}_p = \overline{d}^s_p$, and $\overline{d}^h_p = \overline{d}^s_p$. In fact, $\underline{d}^s_\mu(x) \leq 2^{-s}$ for $x \in E$ a.e. μ^s if $0 < \mu^s(E) < \infty$ and $\overline{d}^s_p(x) \geq 1$ for $x \in E$ a.e. p^s if $0 < p^s(E) < \infty$.

The following theorem shows that under certain conditions a constant $\gamma < 2^{-s}$ exists such that $\underline{d}\mu^s(x) \leq \gamma$ for all x in the Generalized Symmetric Cantor set E. This is a generalization of Theorem 3 [4].

THEOREM 9. *Let E be a generalized Symmetric Cantor set. If the Hausdorff dimension of s is equal to the Packing dimension, then*

$$\underline{d}^s_\mu(x) \leq (\frac{1}{2})^s \left[\frac{m-1}{m^{\frac{1}{s}} - m}\right]^s \quad \textit{for all} \quad x \in E.$$

Proof. Since $\underline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} \leq (\frac{1}{m})^{\frac{1}{s}} < \beta < \frac{1}{m}$, there exist infinitely many n such that $\frac{a_{n+1}}{a_n} < \beta$. Hence, for those same infinitely many (a_{n+1}, a_n),

$$\frac{b_{n+1}}{a_n} = \frac{a_n - m a_{n+1}}{(m-1)a_n} > \frac{1}{m-1}(1 - m\beta).$$

Therefore, $\frac{a_n}{b_{n+1}} < \frac{m-1}{1-m\beta}$. Now, let x be any endpoint of E, and let b_j be the length of the contiguous interval of E with one of the endpoints being x. Then,there exist infinitely many of the above (a_{n+1}, a_n) such that $b_{n+1} + a_{n+1} < b_j$. Let $r = a_{n+1} + b_{n+1}$. Therefore,

$$\begin{aligned}\frac{\mu^s(E \cap B(x,r))}{(2r)^s} &< \frac{(a_{n+1})^s}{(2b_{n+1})^s} = (\frac{1}{2})^s(\frac{a_n}{b_{n+1}})^s(\frac{a_{n+1}}{a_n})^s \\ &< (\frac{1}{2})^s(\frac{m-1}{1-m\beta})^s\beta^s.\end{aligned}$$

Let x be a limit point of E which is not an endpoint of E. Then x is contained in infinitely many closed intervals of length (a_{n+1}, a_n) given above with respect to the endpoints. Therefore $r \geq b_{n+1}$ and

$$\frac{\mu^s(E \cap B(x,r))}{(2r)^s} < (\frac{1}{2})^s(\frac{m-1}{1-m\beta})^s\beta^s.$$

Letting $n \to \infty$ and β approaches $(\frac{1}{2})^s$,

$$\underline{d}_\mu^s(x) \leq (\frac{1}{2})^s(\frac{m-1}{m^{\frac{1}{s}} - m})^s \quad \text{for all} \quad x \in E.$$

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Department of Mathematics
Kyungpook National University
Taegu 702–701, Korea