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SEMILINEAR ABSTRACT CAUCHY PROBLEM ASSOCIATED WITH AN EXPONENTIALLY BOUNDED C–SEMIGROUP IN A BANACH SPACE

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1. Introduction

The purpose of this paper is to consider the initial value problem

$$(1)\qquad \begin{aligned} &\frac{d}{dt}u(t) = Zu(t) + f(t, u(t)), \quad t > 0 \\ &u(0) = u_0 \end{aligned}$$

in a Banach space X, where Z is the generator of an exponentally bounded C-semigroup in X, $f(t,u) : [0,T] \times X \to X$ and $u_0 \in X$.

Davies-Pang [1] showed that (1) with $f(t,u) \equiv 0$ has a unique solution when $T = \infty$ and Ha [2] with $f(t,u) \equiv f(t)$ when $T < \infty$ under some assumptions.

One may refer to Pazy [3] for (1) associated with a C_0 semigroup in X.

In 2, we recall definitions and chracterizations for an exponentially bounded C-semigroup given in [1], [2] which we need. In 3, we are concerned with existence and uniqueness of solutions of (1).

2. Preliminaries

Let X be a Banach space and let C be an injective linear operator from X into itself with dense range $R(C)$ in X. We say that $\{S(t)|t \geq 0\}$ is an exponentially bounded C-semigroup in X if it is a strongly continuous family of bounded linear operator from X into itself satisfying

(a_1) $S(0) = C$,
(a_2) $S(t+s)C = S(t)S(s)$ for $t, s \geq 0$,
(a_3) there exist constants $M \geq 0$ and $a \geq 0$ such that $|S(t)| \leq Me^{at}$ for $t \geq 0$.

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It follows that $S(t)C = CS(t)$ and for $x \in R(C)$, $S(t)x \in R(C)$, $C^{-1}S(t)x = S(t)C^{-1}x$. Let $T(t)$ be the closed linear operator defined by

$$T(t)x = C^{-1}S(t)x \tag{2}$$

for $x \in D(T(t)) = \{x \in X | S(t)x \in R(c)\}$. Then $R(C) \subset D(T(t))$ and

(b_1) $T(0)x = x$ for $x \in X$,
(b_2) $T(t+s)x = T(t)T(s)x$ for $x \in R(C^2)$,
(b_3) $T(t)x$ is continuous in $t \geq 0$ for $x \in R(C^2)$.

Let $\lambda > a$. We define the bounded linear operator L_λ from X into itself by

$$L_\lambda x = \int_0^\infty e^{-\lambda t} S(t) x dt$$

for $x \in X$. Then L_λ with $\lambda > a$ is injective and $(\lambda - L^{-1}C)x$ is independent of $\lambda > a$ for $x \in X$ with $Cx \in R(L_\lambda)$. Set $Zx = (\lambda - L_\lambda^{-1}C)x$ for $x \in D(Z) = \{x \in X | Cx \in R(L_\lambda)\}$ with $\lambda > a$. Then Z is called the generator of $\{S(t) | t \geq 0\}$ with $|S(t)| \leq Me^{at}$ and we have

$$(\lambda - Z)^{-1}Cx = L_\lambda x \text{ and } L_\lambda x \in D(Z) \tag{3}$$

for $x \in X$ and $\lambda > a$. If C is bijective, then $C^k D(Z) = D(Z)$ ($k = 0, 1, 2, \cdots$), where $C^0 = I$ (the identity), $C^k = CC^{k-1}$ and $C^k D(Z) = \{C^k x \in X | x \in D(z)\}$ for $k = 1, 2, \cdots$.

The generator Z is densely defined in X and $S(t)x \in D(Z)$,

$$\frac{d}{dt} S(t)x = ZS(t)x = S(t)Zx \tag{4}$$

for $x \in D(Z)$. Furthermore $T(t)x \in D(Z)$ and

$$\frac{d}{dt} T(t)x = ZT(t)x = T(t)Zx \tag{5}$$

for $x \in CD(Z)$.

DEFINITION 1. A function $u(t) : [0,T] \to X$ is called a solution of (1) on $[0,T]$ if the the following (c_1)-(c_4) are satisfied :

(c_1) $u(t)$ is continuous on $t \in [0,T]$,
(c_2) $u(t)$ is continuously differential in $t \in [0,T]$,
(c_3) $u(t) \in D(Z)$ for $t \in (0,T)$,
(c_4) (1) holds where $T < \infty$.

From the same method as in [2], we have two theorems :

THEOREM 2. *Set $g(t) = f(t,u)$ for every $u \in X$. Let $g(t) \in R(C)$ for $t \in [0,T]$ with $C^{-1}g \in L^1(0,T;X)$. Let g be continuous on $[0,T]$. If $\int_0^t T(t-s)g(s)ds \in C^4D(Z)$ and $Z\int_0^t T(t-s)g(s)ds$ is continuous in $t \in [0,T]$, then (1) has a unique solution on $[0,T]$ with $f(t,u) = g(t)$.*

THEOREM 3. *Set $g(t) = f(t,u)$ for every $u \in X$. Let $g(t) \in R(C^2)$ for $t \in [0,T]$ and let $C^{-1}g(t)$ be continuously differentiable in $t \in [0,T]$ Then (1) has a unique solution on $[0,T]$ with $f(t,u) = g(t)$.*

3. Semilinear abstract Cauchy problem

Throughout this section, let $\{S(t)|t \geq 0\}$ be an exponentially bounded C-semigroup in X with $|S(t)| \leq Me^{at}$ and Z its generator. Let $T(t) = C^{-1}S(t)$ be the operator defined by (2).

We give a property of a solution of (1) on $[0,T]$ by the similar method of the proof in [2].

PROPOSITION 4. *Let $f(t,u) \in R(C^2)$ for $t \in [0,T]$ and $u \in X$ with $C^{-1}f(t,u)$ is continuous in $(t,u) \in [0,T] \times X$. If $u(t)$ is a solution of (1) on $[0,T]$ for $u_0 \in C^2D(Z)$, then*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s)ds \tag{6}$$

for $t \in [0,T]$.

DEFINITION 5. A continuous function $u(t) : [0,T] \times X$ is called a mild solution of (1) on $[0,T]$ if $u(t)$ is satisfied by (6).

THEOREM 6. *Let $f(t,u) \in R(C^2)$ for $t \in [0,T]$ and $u \in X$. Let $C^{-1}f(t,u)$ be continuous in $t \in [0,T]$ and Lipschitz continuous in $u \in X$ with its Lipschitz constant L. Then (1) has a unique mild solution $u(t)$ on $[0,T]$ for $u_0 \in C^2D(Z)$. Moreover, let $u(t), v(t)$ be mild solutions of (1) for $u_0, v_0 \in C^2D(Z)$, respectively. Then*

$$|u(t) - v(t)| \le K|C^{-1}u_0 - C^{-1}v_0|$$

for some constant $K > 0$.

Proof :. Let $u_0 \in C^2D(Z)$. Set $\mathcal{C} = C([0,T];X)$. We define an operator $J : \mathcal{C} \to \mathcal{C}$ by

$$(Ju)(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds$$

for $u \in \mathcal{C}$. Then, every $u, v \in \mathcal{C}$,

$$\begin{aligned}|(Ju)(t) - (Jv)(t)| &\le \int_0^t |S(t)|\,|C^{-1}f(s,u(s)) - C^{-1}f(s,v(s))|ds \\ &\le LMe^{aT}t|u-v|_\infty.\end{aligned}$$

Thus

$$|Ju - Jv|_\infty \le LMe^{at}T|u-v|_\infty,$$

where $|u|_\infty \le \sup_{0\le t\le T}|u(t)|$. Similarly we have for $n \ge 2$ and $u, v \in \mathcal{C}$,

$$|J^n u - J^n v|_\infty \le \frac{(LMe^{aT}T)^n}{n!}|u-v|_\infty.$$

Thus J^n has a fixed point in $\mathcal{C}$ for sufficiently large n such that $\frac{(LMe^{aT}T)^n}{n!} < 1$. Therefore J has a fixed point u in $\mathcal{C}$ and thus $u(t)$ is a mild solution of (1) on $[0,T]$ for $u_0 \in C^2D(Z)$.

Let $u(t), v(t)$ be mild solutions of (1) on $[0,T]$ for $u_0, v_0 \in C^2D(Z)$, respectively. Then

$$|u(t) - v(t)| \le Me^{aT}|C^{-1}u_0 - C^{-1}v_0| + \int_0^t |u(s) - v(s)|ds$$

for $t \in [0,T]$. From Gronwall's inequality, we have (7).

By the similar method as Theorem 6, we have the following result.

PROPOSITION 7. *Under the assumptions as in Theorem 6, the integral equation*

$$u(t) = I(t) + \int_0^t T(t-s)f(s,u(s))ds$$

has a unique solution for every continuous function $I(t)$ on $[0,T]$.

THEOREM 8. *Let $f(t,u) \in R(C^3)$ for $t \in [0,T]$, $u \in X$, and $C^{-3}f(t,u)$ continuously differentiable in $(t,u) \in [0,T] \times X$. Then for every $u_0 \in C^2D(Z)$, a mild solution of (1) is a solution of (1) on $[0,T]$.*

Proof :. Let $u(t)$ be a mild solution of (1) on $[0,T]$ for $u_0 \in C^2D(Z)$. Then

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds. \tag{8}$$

Set $A(t) = \frac{\partial}{\partial u}C^{-1}f(t,u)$,

$$I(t) = T(t)f(0,u_0) + ZT(t)u_0 + \int_0^t T(t-s)\frac{\partial}{\partial s}f(s,u(s))ds$$

and $g(t,u) = A(t)u$ for $t \in [0,T]$ and $u \in X$. Then $g(t,u)$ is continuous in $t \in [0,T]$ and Lipschitz continuous in $u \in X$, and $I(t)$ is continuous in $t \in [0,T]$. Moreover, $g(t,u) \in R(C^2)$ for $(t,u) \in [0,T] \times X$. It follows from Proposition 7 that the integral equation

$$v(t) = I(t) + \int_0^t T(t-s)g(s,v(s))ds$$

has a unique continuous solution on $[0,T]$. From the continuous differentiability of $f(t,u)$, we have

$$f(s,u(s+h)) - f(s,u(s)) = \frac{\partial}{\partial u}f(s,u)(u(s+h)-u(s)) + \epsilon_1(s,h),$$

and

$$f(s+h,u(s+h)) - f(s,u(s+h)) = \frac{\partial}{\partial s}f(s,u(s+h))h + \epsilon_2(s,h),$$

where $\frac{1}{h}|\epsilon_i(s,h)| \to 0$ as $h \to 0$ uniformly on $[0,T]$ for $i = 1,2$. Set

$$v_h(t) = \frac{1}{h}(u(t+h) - u(t)) - v(t).$$

Then

$$v_h(t) = \epsilon_h + \int_0^t T(t-s)g(s, v_h(s))ds, \tag{9}$$

where

$$\epsilon_h = \epsilon_h^{(1)} + \epsilon_h^{(2)} + \epsilon_h^{(3)} + \epsilon_h^{(4)},$$
$$\epsilon_h^{(1)} = \frac{1}{h}(T(t+h)u_0 - T(t)u_0) - ZT(t)u_0,$$
$$\epsilon_h^{(2)} = \frac{1}{h}\int_0^t T(t-s)(\epsilon_1(s,h) + \epsilon_2(s,h))ds,$$
$$\epsilon_h^{(3)} = \int_0^t T(t-s)(\frac{\partial}{\partial s}f(s, u(s+h)) - \frac{\partial}{\partial s}f(s,u(s))ds,$$

and

$$\epsilon_h^{(4)} = \frac{1}{h}\int_0^h T(t+h-s)f(s,u(s))ds - T(t)f(0,u_0).$$

It follows that $\lim_{h\to 0}\epsilon_h^{(i)} = 0$ for $i = 1,2,3,4$ and thus $\lim_{h\to 0}\epsilon_h = 0$. Set

$$K = \max\{|C^{-1}A(s)| \; |0 \le s \le T\}.$$

Then from (9),

$$|v_h(t)| \le |\epsilon_h| + KMe^{aT}\int_0^t |v_h(s)|ds.$$

By Gronwall's inequality,

$$|v_h(t)| \le |\epsilon_h| e^{KTMe^{aT}}$$

and thus $\lim_{h\to 0} v_h(t) = 0$. Therefore $u(t)$ is differentiable on $[0,T]$ and $\frac{du}{dt} = v(t)$. Since $v(t)$ is continuous in $t \in [0,T]$, $u(t)$ is continuously differentiable in $t \in [0,T]$ and $C^{-1}f(t,u(t)$ is also continuously differentiable. From Theorem 3, the initial problem

$$\begin{cases} \frac{d}{dt}w(t) = Zw(t) + f(t,u(t)), \\ w(0) = u_0 \end{cases}$$

has a unique solution $w(t)$ on $[0,T]$ satisfying

$$w(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds. \tag{10}$$

From (8) and (10), $u(t) = w(t)$ on $[0,T]$. Therefore $u(t)$ is a solution of (1) on $[0,T]$.

For an application, let C be a bijective bounded linear operator from X onto itself. Let $\{S(t)|t \geq 0\}$ be an exponentially bounded C-semigroup in X with $|S(t)| \leq Me^{at}$.

$D(Z)$ with a norm $|\cdot|_Z$ defined by $|u|_Z = |u| + |Zu|$ for every $u \in D(Z)$ is a Banach space. Let C_Z, $S_Z(t)$ be the restrictions of $C, S(t)$ on $D(Z)$, respectively. Since $C^kD(Z) = D(Z)$ for $k = 1, 2, \cdots$ and $S(t)u \in D(Z)$ for every $u \in D(Z)$, C_Z is a bijective bounded linear operator from $D(Z)$ onto itself and $\{S_Z(t)|t \geq 0\}$ is an exponentially bounded C_Z-semigroup in $D(Z)$ with $|S_Z(t)|_Z \leq Me^{at}$.

THEOREM 9. *Let $C^{-1}f(t,u) : [0,T] \times D(Z) \to D(Z)$ be continuous in t and Lipschitz continuous in u. Then (1) has a unique solution on $[0,T]$ for every $u_0 \in D(Z)$.*

Proof :. From Theorem 6, (1) has a unique mild solution $u(t)$ on $[0,T]$ for every $u_0 \in D(Z)$ satisfying

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds$$

in $D(Z)$. Since $f(t,u(t))$ is continuous in $t \in [0,T]$ in $D(Z)$, $Zf(t,u(t))$ is also continuous in $t \in [0,T]$ in X. Thus

$$\int_0^t T(t-s)f(s,u(s))ds \in C^4D(Z)$$

and $Z\int_0^t T(t-s)f(s,u(s))ds$ is continuous in $t\in[0,T]$ in X. From Theorem 2, (1) has a unique solution on $[0,T]$ for every $u_0\in D(Z)$.

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