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COMPACT REAL HYPERSURFACES WITH PARALLELY CYCLIC CONDITION OF A COMPLEX PROJECTIVE SPACE

YONG-SOO PYO

1. Introduction

Let $P_n(C)$ be an n-dimensional complex projective space with Fubini Study metric of constant holomorphic sectional curvature 4. In Takagi's study [8] of real hypersurfaces of $P_n(C)$, he proved that all homogeneous real hypersurfaces could be divided into six types which are said to be type A_1, A_2, B, C, D, and E.

In what follows an induced almost contact metric structure of a real hypersurface M of $P_n(C)$ is denoted by (ϕ, g, ξ, η). The structure vector ξ is said to be principal if $A\xi = \alpha\xi$, where A is the shape operator in the direction of the unit normal on M and $\alpha = \eta(A\xi)$. Real hypersurfaces of $P_n(C)$ have been studied by many differential geometers. ([1], [3], [4], [5], [6], [7], and [8] etc.) And one of them, Okumura [7] showed that M is of type A_1 or A_2 if and only if $A\phi = \phi A$. Furthermore, Maeda [6] proved that M is of type A_1 or A_2 if and only if

$$g((\nabla_X A)Y, Z) + \eta(Y)g(\phi X, Z) + \eta(Z)g(\phi X, Y) = 0$$

for any vector fields X, Y, and Z on M, where ∇ is the Riemannian connection with respect to g.

In this paper, we shall prove the following theorem.

THEOREM. *Let M be a compact real hypersurface with parallely cyclic condition of a complex projective space $P_n(C)$. Then M is locally congruent to one of the homogeneous hypersurfaces of type A_1 or A_2.*

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2. Preliminaries

Let M be a real hypersurface of a complex projective space $P_n(C)$. Throughout the present paper the following convention on the range of indices are used, unless otherwise stated

$$i, j, \cdots = 1, 2, \cdots, 2n-1.$$

The summation convention will be used with respect to those system of indices.

For an almost contact metric structure (ϕ, g, ξ, η) on M, the following relations are given :

$$\begin{aligned} \phi_j^r \phi_r^h &= -\delta_j^h + \xi_j \xi^h, \qquad \phi_{jr}\xi^r = 0, \\ \xi_r \phi_j^r &= 0, \qquad \xi_j \xi^j = 1. \end{aligned} \tag{2.1}$$

Furthermore, the covariant derivative of the structure tensors are obtained by

$$\nabla_j \phi_i^h = -h_{ji}\xi^h + h_j^h \xi_i, \qquad \nabla_j \xi_i = -h_{jr}\phi_i^r, \tag{2.2}$$

where ∇ is the Riemannian connection with respect to g and $A = (h_{ji})$ denotes the shape operator with respect to the unit normal on M. Since $P_n(C)$ is of constant holomorphic sectional curvature 4, the Gauss and Codazzi equations are respectively given as follows :

$$\begin{aligned} R_{kjih} = g_{kh}g_{ji} - g_{jh}g_{ki} + \phi_{kh}\phi_{ji} - \phi_{ki}\phi_{jh} \\ - 2\phi_{kj}\phi_{ih} + h_{kh}h_{ji} - h_{ki}h_{jh}, \end{aligned} \tag{2.3}$$

$$\nabla_k h_{ji} - \nabla_j h_{ki} = \xi_k \phi_{ji} - \xi_j \phi_{ki} - 2\xi_i \phi_{kj}, \tag{2.4}$$

where R_{kjih} are the components of the Riemannian curvature tensor of M. Let S_{ji} be the components of the Ricci tensor of M. Then the Gauss equation implies

$$S_{ji} = (2n+1)g_{ji} - 3\xi_j \xi_i + h h_{ji} - h_{ji}^2, \tag{2.5}$$

where h is the trace of the shape operator A and $h_{ji}^2 = h_{jr}h_i^r$.

3. Proof of the Theorem

Let M be a real hypersurface of a complex projective space $P_n(C)$. Then M is called parallely cyclic if $\nabla_m T_{kji} = 3a\nabla_m U_{kji}$ (a is constant), where $T_{kji} = \nabla_k h_{ji} + \nabla_j h_{ki} + \nabla_i h_{jk}$, $U_{kji} = \xi_k \xi_{ji} + \xi_j \xi_{ki} + \xi_i \xi_{jk}$ and $\xi_{ji} = \nabla_j \xi_i + \nabla_i \xi_j$ (see [2]). From now on, we suppose that M is compact and of parallely cyclic.

From (2.2), we have

$$(3.1) \qquad \nabla_k \nabla_j \xi_i = (h_{jr}\xi^r)h_{ki} - h_j k^2 \xi_i - (\nabla_k h_{jr})\phi_i^r.$$

Hence, the Laplacian $\Delta\xi_i$ of ξ_i is given by

$$(3.2) \qquad \Delta\xi_i = h_i r^2 \xi^r - h_2 \xi_i - (\nabla_r h)\phi_i^r,$$

where $h_2 = h_{ji}h^{ji}$. Multiplying ϕ^{ki} to (2.4), by (2.1) and (2.2), we get

$$(3.3) \qquad (\nabla_k h_{ji})\phi^{ki} = -(\phi_{ki}\phi^{ki})\xi_j = -2(n-1)\xi_j.$$

Thus, from (3.1) and (3.3), we obtain

$$(3.4) \qquad \nabla^m \nabla_j \xi_m = hh_{jr}\xi^r - h_j r^2 \xi^r + 2(n-1)\xi_j.$$

Since $\xi_{jm} = \nabla_j \xi_m + \nabla_m \xi_j$, we have

$$(3.5) \qquad \nabla^m \xi_{jm} = hh_{jr}\xi^r - h_2\xi_j - (\nabla_r h)\phi_j^r + 2(n-1)\xi_j$$

by (3.2) and (3.4). Therefore

$$(3.6) \qquad \xi^j \nabla^m \xi_{jm} = -h_2 + 2(n-1),$$

where $\alpha = h_{ji}\xi^j\xi^i$. Since M is of parallely cyclic and

$$(3.7) \qquad \nabla_m T_{kji} = 3\nabla_m \nabla_k h_{ji} - 3\nabla_m(\xi_j \phi_{ik} + \xi_i \phi_{jk})$$

by (2.4), we obtain

$$(3.8) \quad \nabla_m \nabla_k h_{ji} = \nabla_m(\xi_j \phi_{ik} + \xi_i \phi_{jk}) + a\nabla_m(\xi_k \xi_{ji} + \xi_j \xi_{ki} + \xi_i \xi_{jk}).$$

Applying $i = j$ to (3.7) and summing up with respect to j,

$$(3.9) \qquad \nabla_m \nabla_k h = 2a\nabla_m(\xi^r \nabla_r \xi_k).$$

Hence, if we put $W_i = h\nabla_i h - 2ah\xi^r\nabla_r\xi_i + \xi_i(\nabla_r h)\xi^r$, then we get

$$\nabla^i W_i = \|\nabla_i h - 2a\xi^r \nabla_r \xi_i\|^2$$

because $\nabla^i \xi_i = 0$. Since M is compact, we obtain the following lemma by the Stoke's theorem.

LEMMA. *Under the hypothesis of the Theorem, we have*

$$\nabla_i h = 2a\xi^r \nabla_r \xi_i. \tag{3.10}$$

On the other hand, from (3.9), we obtain

$$a\{(\nabla_j \xi^r)(\nabla_r \xi_i) - (\nabla_i \xi^r)(\nabla_r \xi_j) + \xi^r(\nabla_j \nabla_r \xi_j - \nabla_i \nabla_r \xi_j)\} = 0. \tag{3.11}$$

Multiplying ξ^i to (3.11), and using (2.1), (2.2), and (3.1), we can know

$$a(\xi^i \nabla_i \xi^r)(\nabla_r \xi_j) = a\{\beta \xi_j - h^2_{jr}\xi^r + \xi^t \xi^s (\nabla_t h_{sr})\phi_j{}^r\}, \tag{3.12}$$

where $\beta = h^2_{mr}\xi^m \xi^r$. And, by (2.2),

$$(\xi^i \nabla_i \xi^r)(\nabla_j \xi_r) = h^2_{jr}\xi^r - \alpha h_{jr}\xi^r. \tag{3.13}$$

From (2.2), we have

$$\begin{aligned} \nabla_m \xi_{ji} = &-(\nabla_m h_{jr})\phi^r_i - (\nabla_m h_{ir})\phi^r_j + h_{mi}h_{jr}\xi^r \\ &+ h_{mj}h_{ir}\xi^r - h^2_{jm}\xi_i - h^2_{im}\xi_j. \end{aligned} \tag{3.14}$$

Hence, we obtain

$$(\nabla_m \xi_{ji})\xi^m \xi^i = 2\alpha h_{jr}\xi^r - h^2_{jr}\xi^r - \beta\xi_j - \xi^t \xi^s (\nabla_t h_{sr})\phi^r_j. \tag{3.15}$$

From (2.3) and (2.5), we have

$$(S_{kr}h_j{}^r - R_{mkjr}h^{mr})\xi^k = hh_{jr}{}^2\xi^r + \{(2n+1) - h_2\}h_{jr}\xi^r - h\xi_j. \tag{3.16}$$

And, using the Ricci identity, we find

$$\begin{aligned} (S_{kr}h_j{}^r - R_{mkjr}h^{mr})\xi^k = &\, 2\alpha\xi_j - h_{jr}\xi^r - h\xi_j \\ &- a\{2\xi^r(\nabla_r \xi_m)(\nabla_j \xi^m) + \xi^r(\nabla_r \xi_m)(\nabla^m \xi_j) \\ &- \nabla^m \xi_{jm} + \xi^k \xi^m \nabla_k \xi_{jm} - \xi^k \xi_j \nabla^m \xi_{km}\} \end{aligned} \tag{3.17}$$

by (2.1), (2.2) and (3.8). Combining (3.16) and (3.17), and using (2.1), (2.2), (3.5), (3.6), (3.12), (3.13), and (3.15),

(3.18)
$$\begin{aligned} &h h_{jr}{}^2 \xi^r + (2n - h_2) h_{jr} \xi^r - 2\alpha \xi_j \\ &\qquad = a[h h_{jr} \xi^r + \{h\alpha + 4(n-1) - 2h_2\} \xi_j - (\nabla_r h) \phi_j{}^r]. \end{aligned}$$

Since $\nabla_r h = -2a\xi^t h_{ts} \phi_r^s$ by (2.2) and (3.10),

$$(3.19) \qquad (\nabla_r h)\phi_j{}^r = 2a(h_{jr}\xi^r - \alpha\xi_j).$$

Hence, multiplying ξ^j to (3.18), we have

$$(3.20) \qquad h(\beta - 2a\alpha) = (\alpha - 2a)\{h_2 - 2(n-1)\}.$$

On the other hand, from (3.5) and (3.19), we get

$$(3.21) \qquad \nabla^m \xi_{jm} = (h - 2a) h_{jr} \xi^r + \{2a\alpha - h_2 + 2(n-1)\} \xi_j.$$

Thus, we have

$$(3.22) \qquad h^{ji} \xi_i \nabla^m \xi_{jm} = \beta(h - 2a) + \alpha\{2a\alpha - h_2 + 2(n-1)\}.$$

From (2.1), (2.4), and (3.14), we obtain

$$\begin{aligned} (3.23) \qquad h^{ji} \xi_m \nabla^m \xi_{ji} &= 2\xi^m (\nabla_m h_{ji})(\nabla^j \xi^i) \\ &= 2\xi^m (\nabla_j h_{im})(\nabla^j \xi^i) + 2(h - \alpha). \end{aligned}$$

Since, from (3.1) and (3.3),

$$(3.24) \qquad \nabla_j(\xi^r \nabla_r \xi^j) = (\nabla_j \xi^i)(\nabla_i \xi^j) + h\alpha - \beta + 2(n-1),$$

(3.25)
$$\begin{aligned} \|\xi_{ji}\|^2 &= 2(\nabla_j \xi^i)(\nabla^j \xi_i) + 2(\nabla_j \xi^i)(\nabla_i \xi^j) \\ &= 2\nabla_j(\xi^r \nabla_r \xi^j) + 2\{h_2 - h\alpha - 2(n-1)\} \end{aligned}$$

by (2.1) and (2.2).

From (2.2) and (3.8), the Laplacian $\triangle h_{ji}$ of h_{ji} is given by

(3.26)
$$\begin{aligned}\triangle h_{ji} = &(\nabla^m \xi_j)\phi_{im} + (\nabla^m \xi_i)\phi_{jm} + 2h\xi_j\xi_i \\ &- h_{ir}\xi^r\xi_j - h_{jr}\xi^r\xi_i + a\{(\nabla^m \xi_j)\xi_{im} + (\nabla^m \xi_i)\xi_{jm} \\ &+ \xi_m \nabla^m \xi_{ji} + \xi_j \nabla^m \xi_{im} + \xi_i \nabla^m \xi_{jm}\}.\end{aligned}$$

Hence, by (2.1), (2.2), (3.14) and (3.21), we have

(3.27) $$(\triangle h_{ji})\xi^j\xi^i = 2(h-\alpha) - 2a\{h_2 - h\alpha - 2(n-1) + \beta - \alpha^2\}.$$

From (3.23), (3.25), and (3.27), we get

(3.28)
$$\begin{aligned}&\triangle\alpha - 2\nabla^m(h_{ji}\xi^j\nabla_m\xi^i) - 2a\nabla_j(\xi^r\nabla_r\xi^j) \\ &= (\triangle h_{ji})\xi^j\xi^i + 2\xi^m(\nabla_j h_{im})(\nabla^j\xi^i) - 2a\nabla_j(\xi^r\nabla_r\xi^j) \\ &= 2\xi^m(\nabla_m h_{ji})(\nabla^j\xi^i) - a\|\xi_{ji}\|^2 - 2a(\beta - \alpha^2).\end{aligned}$$

Multiplying h^{ji} to (3.26), and using (2.2), (3.22), (3.23), and (3.24),

$$\begin{aligned}h^{ji}\triangle h_{ji} = &2\nabla_j(\xi^r\nabla_r\xi^j) + 2a\xi^m(\nabla_m h_{ji})(\nabla^j\xi^i) \\ &+ 2a[\beta(h-2a) + \alpha\{2a\alpha - h_2 + 2(n-1)\}] - 4(n-1)\end{aligned}$$

because $h^{ji}(\nabla^m\xi_j)\xi_{im} = 0$. Hence, from (3.20) and (3.25), we have

$$\begin{aligned}&h^{ji}\triangle h_{ji} - 2(2a^2+1)\nabla_j(\xi^r\nabla_r\xi^j) \\ &= 2a\xi^m(\nabla_m h_{ji})(\nabla^j\xi^i) - 2a^2\|\xi_{ji}\|^2 - 4a^2(\beta-\alpha^2) - 4(n-1).\end{aligned}$$

Since $\frac{1}{2}\triangle h_2 = h^{ji}\triangle h_{ji} + \|\nabla_k h_{ji}\|^2$, we obtain

(3.29) $$\begin{aligned}\triangle F = &\|\nabla_k h_{ji}\|^2 + 2a\xi^m(\nabla_m h_{ji})(\nabla^j\xi^i) \\ &- 2a^2\|\xi_{ji}\|^2 - 4a^2(\beta-\alpha^2) - 4(n-1),\end{aligned}$$

where $\triangle F = \frac{1}{2}\triangle h_2 - 2(2a^2+1)\nabla_j(\xi^r\nabla_r\xi^j)$. If we put

$$\nabla_k h^*_{ji} = \nabla_k h_{ji} - \xi_j \phi_{ik} - \xi_i \phi_{jk} - a(\xi_j \xi_{ik} + \xi_i \xi_{jk} + \xi_k \xi_{ij}),$$

then we get

$$\begin{aligned}\|\nabla_k h^*_{ji}\|^2 = \|\nabla_k h_{ji}\|^2 - 12a\xi^k(\nabla_k h_{ji})(\nabla^j \xi^i) \\ + 3a^2\|\xi_{ji}\|^2 + 6a^2(\beta - \alpha^2) - 4(n-1)\end{aligned}$$

by (2.1), (2.2), (3.3), (3.13) and (3.23). Thus, by (3.29),

$$\triangle F = \|\nabla_k h^*_{ji}\|^2 + 14a\xi^k(\nabla_k h_{ji})(\nabla^j \xi^i) - 5a^2\|\xi_{ji}\|^2 - 10a^2(\beta - \alpha^2).$$

Since M is compact and $(\nabla_r \xi^j)(\nabla_s \xi_j)\xi^r \xi^s = \beta - \alpha^2 \geq 0$, from (3.28), we have

$$\|\nabla_k h^*_{ji}\|^2 + 2a^2\|\xi_{ji}\|^2 + 4a^2(\beta - \alpha^2) = 0$$

by the Stoke's theorem. Consequently, we obtain $a\|\xi_{ji}\| = 0$. If $\|\xi_{ji}\| = 0$, then, from (2.2), we get $h_{jr}\phi_i{}^r + h_{ir}\phi_j{}^r = 0$. And, if $a = 0$, then $\nabla_k h_{ji} = \xi_j \phi_{ik} + \xi_i \phi_{jk}$ because $\nabla_k h^*_{ji} = 0$. Therefore, M is of type A_1 or A_2 (see [6] and [7]).

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Department of Applied Mathematics
National Fisheries University of Pusan
Pusan 608–737, Korea