Pusan Kyŏngnam Math. J. 7(1991), No. 2, pp. 119–128

SHARP FUNCTION ESTIMATES FOR A CLASS OF PSEUDODIFFERENTIAL OPERATORS

H. S. KIM** AND S. S. SHIN*

1. Introduction

We say that a symbol $a(x,\xi)$ in the class $S^m_{\rho,\delta}$, or that $a \in S^m_{\rho,\delta}$, if for x, ξ in $\mathbf{R}^n$ and multiindices α, β

$$|(\frac{\partial}{\partial x})^{\alpha}(\frac{\partial}{\partial \xi})^{\beta}| \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|}$$

From the definition, it is obvious that $S^{m_1}_{\rho_1,\delta_1} \supset S^{m_2}_{\rho_2,\delta_2}$ if $\rho_1 \le \rho_2$, $\delta_1 \ge \delta_2$ and $m_1 \ge m_2$.

If $a(x,\xi)$ is a symbol in $S^m_{\rho,\delta}$, then it defines a pseudodifferential operator A, by formula;

$$Af(x) = \int_{\mathbf{R}^n} a(x,\xi)\hat{f}(\xi)e^{2\pi i x\,\xi}d\xi$$

where f is a Schwartz function and $\hat{f}$ denotes the Fourier transform of f.

For a locally integrable function f , we define the sharp function $f^\sharp(x)$ by formula

$$f^\sharp(x) = \sup_Q \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy$$

the supremum is taken over all cubes Q containing x, and f_Q is the average value of f on the cube Q.

Received October 27, 1991.

* This research is supported by the Basic Science Research Institute, Ministry of Education,1991.

** This research is supported by TGRC–KOSEF,1991.

For $1 \le p < \infty$,

$$M_p f(x) = \sup_Q \{\frac{1}{|Q|}\int_Q |f(y)|^p dy\}^{\frac{1}{p}}$$

the supremum being taken over all cubes Q containing x.

Now for a symbol $a(x,\xi) \in S^0_{1,0}$, it has shown in Miller [5] that for each $1 < p < \infty$,

$$(Af)^\sharp(x) \le CM_p f(x), \quad x \in \mathbf{R}^n \quad \text{and} \quad f \in C_0^\infty(\mathbf{R}^n).$$

Also Chanillo and Torchinsky [2] showed that for a symbol $a(x,\xi) \in S^{-n(1-\rho)/2}_{\rho,\delta}$, $0 \le \delta < \rho < 1$,

$$(Af)^\sharp(x) \le CM_2 f(x), \quad x \in \mathbf{R}^n \quad \text{and} \quad f \in C_0^\infty(\mathbf{R}^n).$$

It is well-known that these sharp function estimates lead to various weighted L^p inequalities for Muckenhoupt's A_p weights.

We know that $\psi \in S^a_{1,0}$ and is real valued; it follows that $e^{i\psi} \in S^0_{1-a,a}$, $0 < a < 1$. Consequently we are interested in the case $0 < \rho < \delta \le 1$. The well-known L^2 results for pseudodifferential operator with symbol $a(x,\xi) \in S^m_{\rho,\delta}$ with $0 < \rho \le \delta \le 1$ have been established by L.Rodino [6]. So the purpose of this paper is to prove the sharp function estimates for pseudodifferential operators with symbols $a(x,\xi)$ in $S^m_{\rho,\delta}$ with $0 < \rho \le \delta \le 1$.

Now we state our theorems.

THEOREM 1. *Let A be a pseudodifferential operator with symbol $a(x,\xi) \in S^{\frac{n}{2}(2\rho-\delta-1)}_{\rho,\delta}$ with $0 < \rho \le \delta < 1$. Then for $f \in C_0^\infty(\mathbf{R}^n)$, $(Af)^\sharp(x) \le CM_2 f(x)$ for all $x \in \mathbf{R}^n$.*

THEOREM 2. *Let A be a pseudodifferential operator with symbol $a(x,\xi) \in S^{m-\epsilon}_{\rho,1}$, $m = (\frac{n}{2}+1)(\rho-1)$, $0 < \rho < 1$ and $\epsilon > 0$. Then for $f \in C_0^\infty(\mathbf{R}^n)$, $(Af)^\sharp(x) \le CM_2 f(x)$ for all $x \in \mathbf{R}^n$.*

2. Preliminary Lemmas

We begin by introducing a notation. By $|x| \sim t$ we denote the fact that the values of x in question lie in the annulus $\{x \in \mathbf{R}^n : t < |x| < 2t\}$.

LEMMA 2.1. [2]. *Let* $a(x,\xi) \in S^{\frac{n}{2}(\rho-1)}_{\rho,\delta}$, $0 < \delta \le 1$, $0 < \rho < 1$. *Let* $K(x,\omega)$ *denote the inverse Fourier transform, in the* ξ*-variable and in the distribution sense of* $a(x,\xi)$, *that is fomally*

$$K(x,\omega) = \int_{\mathbf{R}^n} e^{2\pi i\xi\cdot\omega} a(x,\xi)d\xi.$$

Then for $|x - x_0| \le d < \frac{1}{2}$ *and* $N \ge 1$,

$$\left(\int_{|y-x_0|\sim(2^N d)^\rho} |K(x,x-y) - K(x_0,x_0-y)|^2 dy\right)^{\frac{1}{2}} \le \frac{C|x-x_0|^{\rho(l-\frac{n}{2})}}{(2^N d)^{l\rho}}$$

where l *is an integer such that* $\frac{n}{2} < l < \frac{n}{2} + \frac{1}{\rho}$.

LEMMA 2.2. *Let* $a(x,\xi) \in S^{m-\epsilon}_{\rho,1}$, $m = (\frac{n}{2}+1)(\rho-1)$, $\epsilon > 0$ *and* $0 < \rho < 1$. *Let* $K(x,\omega)$ *denote the inverse Fourier transform, in the* ξ*-variable and in the distribution sense of* $a(x,\xi)$, *that is fomally*

$$K(x,\omega) = \int_{\mathbf{R}^n} e^{2\pi i\xi\cdot\omega} a(x,\xi)d\xi.$$

Then for $|x - x_0| \le d < \frac{1}{2}$ *and* $N \ge 1$,

$$\left[\int_{|y-x_0|\sim 2^N d} |K(x,x-y) - K(x_0,x_0-y)|^2 dy\right]^{\frac{1}{2}} \le \frac{C|x-x_0|^{-m+l\rho-\frac{n}{2}}}{(2^N d)^l}$$

where l *is an integer with* $\frac{n}{2} < l \le \frac{n}{2} + 1$.

Proof. Let $\sum_{j\ge 0}\theta_j(\xi) = 1$ be a partition of unity such that $\theta_j(\xi)$ is supported in $|\xi| \sim 2^j, j \ge 1$, and $\theta_0(\xi)$ is supported in $|\xi| \le 2$. Let

$$a(x,\xi) = \sum_{j\ge 0}\theta_j(\xi)a(x,\xi) = \sum_{j\ge 0} a_j(x,\xi).$$

Moreover, put

$$\int_{\mathbf{R}^n} e^{2\pi i x\cdot\xi} a_j(x,\xi)\hat{f}(\xi)d\xi = \int_{\mathbf{R}^n} K_j(x, x-y)f(y)dy, j \geq 0,$$

where

$$K_j(x, x-y) = \int e^{2\pi i\xi\cdot(x-y)} a_j(x,\xi)d\xi.$$

Thus $K(x,\omega) = \sum_{j\geq 0} K_j(x,\omega)$. We now get

$$\left[\int_{|y-x_0|\sim 2^N d} |K(x,x-y) - K(x_0,x_0-y)|^2 dy\right]^{\frac{1}{2}}$$
$$\leq \sum_{j\geq 0}\left[\int_{|y-x_0|\sim 2^N d} |K_j(x,x-y) - K_j(x_0,x_0-y)|^2 dy\right]^{\frac{1}{2}}.$$

We choose j_0 such that $2^{j_0}|x-x_0| \sim 1$, and break up the sum on the above as follows:

$$\sum_{j<j_0}(\int_{|y-x_0|\sim 2^N d} |K_j(x,x-y) - K_j(x_0,x_0-y)|^2 dy)^{\frac{1}{2}}$$
$$+ \sum_{j\geq j_0}(\int_{|y-x_0|\sim 2^N d} |K_j(x_0,x_0-y)|^2 dy)^{\frac{1}{2}}$$
$$+ \sum_{j\geq j_0}(\int_{|y-x_0|\sim 2^N d} |K_j(x,x-y)|^2 dy)^{\frac{1}{2}} = I + II + III, \quad \text{say}.$$

We now consider I.

(2.1)

$$|I| \leq \sum_{j<j_0}\left[\int_{|y-x_0|\sim 2^N d} |K_j(x,x-y) - K_j(x_0,x-y)|^2 dy\right]^{\frac{1}{2}}$$
$$+ \sum_{j<j_0}\left[\int_{|y-x_0|\sim 2^N d} |K_j(x_0,x-y) - K_j(x_0,x_0-y)|^2 dy\right]^{\frac{1}{2}}$$

$$= I_1 + I_2, \quad \text{say.}$$

Now, by Hausdorff Young's inequality,

$$\begin{aligned}
|I_1| &\leq \sum_{j<j_0} \left[\int_{|y-x_0|\sim 2^N d} |K_j(x,x-y) - K_j(x_0,x-y)|^2 dy \right]^{\frac{1}{2}} \\
&\leq C \sum_{j<j_0} (2^N d)^{-l} \\
&\quad \cdot \left[\int_{|y-x_0|\sim 2^N d} (|K_j(x,x-y) - K_j(x_0,x-y)| \cdot |y-x|^l)^2 dy \right]^{\frac{1}{2}} \\
&\leq C \sum_{j<j_0} (2^N d)^{-l} \sum_{|\beta|=l} \left[\int \sup_\eta |(\frac{\partial}{\partial \eta})(\frac{\partial}{\partial \xi})^\beta a_j(\eta,\xi)|^2 d\xi \right]^{\frac{1}{2}} |x - x_0| \\
&\leq C \sum_{j<j_0} (2^N d)^{-l} 2^{j[m-\epsilon-l\rho+1+\frac{n}{2}]} |x-x_0| \\
&\leq C (2^N d)^{-l} 2^{j_0[m-l\rho+1+\frac{n}{2}]} |x - x_0| \sum_{j<j_0} 2^{-j\epsilon}.
\end{aligned}$$

Since $l \leq \frac{n}{2} + 1$ and $2^{j_0}|x - x_0| \sim 1$, the above sum is dominated by

$$C(2^N d)^{-l} 2^{j_0(m-l\rho+\frac{n}{2})} \leq C(2^N d)^{-l} |x - x_0|^{-m+l\rho-\frac{n}{2}} \tag{2.2}$$

We consider the second term I_2 in (2.1). In this case, we first dominate this term by

$$\sum_{j<j_0} (2^N d)^{-l} \cdot \left[\int_{|y-x_0|\sim 2^N d} (|y-x_0|^l |K_j(x_0, x-y) - K_j(x_0, x_0 - y)|)^2 dy \right]^{\frac{1}{2}}. \tag{2.3}$$

Now we consider the integrand, for $|\alpha| = l$,

$$\begin{aligned}
&|y - x_0|^l \cdot |K_j(x_0, x - y) - K_j(x_0, x_0 - y)| \\
&= |y - x_0|^l \left| \int e^{2\pi i(x_0 - y)\cdot\xi} a_j(x_0, \xi)(e^{2\pi i(x - x_0)\cdot\xi} - 1) d\xi \right| \\
&= C \left| \int e^{2\pi i(x_0 - y)\cdot\xi} (\frac{\partial}{\partial\xi})^\alpha [a_j(x_0, \xi)(e^{2\pi i(x - x_0)\cdot\xi} - 1)] d\xi \right|.
\end{aligned}$$

So by Hausdorff Young inequality and Leibniz's formular, we can majorize (2.3) by

$$\begin{aligned}
&C \sum_{j<j_0} (2^N d)^{-l} \\
&\quad \cdot \left(\int \sum_{|\beta|+|\gamma|=l} |(\frac{\partial}{\partial\xi})^\beta a_j(x_0, \xi)(\frac{\partial}{\partial\xi})^\gamma (e^{2\pi i(x - x_0)\cdot\xi} - 1)|^2 d\xi \right)^{\frac{1}{2}} \\
&\le C \sum_{j<j_0} (2^N d)^{-l} \sum_{\substack{|\beta|+|\gamma|=l \\ |\gamma|\neq 0}} |x - x_0|^{|\gamma|} 2^{j(m-\epsilon-\rho|\beta|+\frac{n}{2})} \\
&\quad + C \sum_{j<j_0} (2^N d)^{-l} |x - x_0| 2^{j(m-\epsilon-\rho l+\frac{n}{2}+1)}.
\end{aligned}$$

But $|x - x_0| \le \frac{1}{2}$ and since $l \le \frac{n}{2} + 1$, the second sum above dominates the first one. Since $2^{j_0}|x - x_0| \sim 1$, this second term is bounded by $C(2^N d)^{-l}|x - x_0|^{-m+l\rho-\frac{n}{2}}$. Similarly, we get

$$\begin{aligned}
|II| &\le C(2^N d)^{-l}|x - x_0|^{-m+l\rho-\frac{n}{2}} \quad \text{and,} \\
|III| &\le C(2^N d)^{-l}|x - x_0|^{-m+l\rho-\frac{n}{2}}.
\end{aligned}$$

This completes the proof.

LEMMA 2.3. *If $a \in S^{m_1}_{\rho,\delta}$ and $b \in S^{m_2}_{\rho,\delta}$, $0 \le \rho$, $\delta \le 1$, then $ab \in S^{m_1+m_2}_{\rho,\delta}$.*

Proof. See Proposition 3.4 [1].

3. The Proofs of the Main Theorems

The proof of Theorem 1: Let $f \in C_0^\infty(\mathbf{R}^n)$. Given $x_0 \in \mathbf{R}^n$ and a cube Q centered at x_0 of side length d. The non-trivial case is when $d \leq 1$, which we consider.

Let $f(x) = f_1(x) + f_2(x)$ with $f_1(x) = f(x)\chi_B(x)$, where B is a cube concentric with Q of side length d^ρ. Let $a(x,\xi) = a(x,\xi)\ |\xi|^{n(1-\rho)/2}$ $|\xi|^{n(\rho-1)/2} = q(x,\xi)\ |\xi|^{n(\rho-1)/2}$, say. We note that, by Lemma 2.3, $q(x,\xi) \in S_{\rho,\delta}^{\frac{n}{2}(\rho-\delta)}$. So the pseudodifferential operator with symbol $q(x,\xi)$ is bounded on $L^2(\mathbf{R}^n)$ [6]. We denote this operator by G.

Let $\frac{1}{p} = \frac{1}{2} - \frac{(1-\rho)}{2}$. Then by the usual Hardy-Littlewood-Sobolev fractional integration theorem and the L^2-boundedness of G we get, $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\begin{aligned}\int_Q |Af_1(x)|dx &\leq Cd^{n/p'}\Big(\int_{\mathbf{R}^n} |Af_1(x)|^p dx\Big)^{\frac{1}{p}} \\ &\leq Cd^{\frac{n}{p'}}\Big(\int_{\mathbf{R}^n} |Gf_1(x)|^2 dx\Big)^{\frac{1}{2}} \\ &\leq Cd^{\frac{n}{p'}}\Big(\int_B |f_1(x)|^2 dx\Big)^{\frac{1}{2}} \\ &\leq Cd^n M_2 f_1(x_0).\end{aligned}$$

We now estimate the term involving $Af_2(x)$. Let

$$Af_2(x) = \int_{\mathbf{R}^n} K(x, x-y) f_2(y)dy,$$

and let

$$C_I = \int_{\mathbf{R}^n} K(x_0, x_0 - y) f(y)dy$$

where $K(x,\omega) = \int\limits_{\mathbf{R}^n} e^{2\pi i\xi\cdot\omega} a(x,\xi)d\xi$.

Then we get, for $x \in Q$,

$$|Af_2(x) - C_I| \leq \int_{\mathbf{R}^n} |K(x, x-y) - K(x_0, x_0 - y)||f_2(y)|dy$$
$$\sum_{N=1}^{\infty} \Big(\int_{|y-x_0| \sim (2^N d)^\rho} |K(x, x-y) - K(x_0, x_0 - y)|^2 dy \Big)^{\frac{1}{2}}$$
$$\times \Big(\int_{|y-x_0| \sim (2^N d)^\rho} |f_2(y)|^2 dy \Big)^{\frac{1}{2}}.$$

Since $S^{\frac{n}{2}}_{\rho,\delta}(2\rho - \delta - 1) \subseteq S^{\frac{n}{2}(\rho-1)}_{\rho,\delta}$, for $\rho \leq \delta$, applying Lemma 1 to the first term in the summands on the right above. Hence we get, for $|x - x_0| \leq d$,

$$|Af_2(x) - C_I| \leq C \sum_{N=1}^{\infty} |x - x_0|^{\rho(l - \frac{n}{2})} (2^N d)^{-l\rho + \frac{n}{2}\rho} M_2 f(x_0)$$
$$\leq C \sum_{N=1}^{\infty} (2^N)^{\rho(\frac{n}{2} - l)} M_2 f(x_0) \leq C M_2 f(x_0)$$

because $l > \frac{n}{2}$. This concludes the case $d \leq 1$.

In the case $d > 1$, we easily get desired result following closely a proof of Chanillo and Torchinski [2].

The proof of Theorem 2: Let $f \in C_0^\infty(\mathbf{R}^n)$. Given $x_0 \in \mathbf{R}^n$ and a cube Q centered at x_0 with diameter d. The nontrivial case is when $d \leq 1$, which we consider. Let $f(x) = f_1(x) + f_2(x)$ with $f_1(x) = f(x)\chi_I(x)$, where $I = \{y \in \mathbf{R}^n : |y - x_0| \leq 3d\}$. Then, for $x \in Q$,

$$\frac{1}{|Q|} \int_Q |Af_1(x)| dx \leq \left(\frac{1}{|Q|} \int_Q |Af_1(x)|^2 dx \right)^{\frac{1}{2}}$$
$$\leq \left(\frac{1}{|Q|} \int_{\mathbf{R}^n} |Af_1(x)|^2 \right)^{\frac{1}{2}}$$

Since A is bounded on $L^2(\mathbf{R}^n)$, the above term dominated by

$$\leq C\frac{1}{|Q|^{\frac{1}{2}}}\left(\int_{\mathbf{R}^n}|f_1(x)|^2dx\right)^{\frac{1}{2}}$$
$$= C\left(\frac{1}{|Q|}\int_I |f_1(x)|^2dx\right)^{\frac{1}{2}}$$
$$\leq CM_2f_1(x_0).$$

We now estimates the term involving $Af_2(x)$. Since

$$Af_2(x) = \int_{\mathbf{R}^n} K(x, x-y)f_2(y)dy,$$

letting $C_I = \int_{\mathbf{R}^n} K(x_0, x_0 - y)f_2(y)dy$.

We get, for $x \in Q$, by Hölder inequality,

$$|Af_2(x) - C_I|$$
$$\leq \int_{\mathbf{R}^n} |K(x, x-y) - K(x_0, x_0 - y)| \cdot |f_2(y)|dy$$
$$\leq \sum_{N=1}^{\infty}\left[\int_{|y-x_0|\sim 2^N d} |K(x, x-y) - K(x_0, x_0 - y)|^2 dy\right]^{\frac{1}{2}}$$
$$\cdot \left[\int_{|y-x_0|\sim 2^N d} |f_2(x)|^2 dy\right]^{\frac{1}{2}}.$$

By Lemma 2.2., the above term is dominated by

$$C\sum_{N=1}^{\infty}(2^N d)^{-l}|x - x_0|^{-m+l\rho-\frac{n}{2}}(2^N d)^{\frac{n}{2}}Mf_2(x_0),$$

where $\frac{n}{2} < l \leq \frac{n}{2} + 1$. Since $-m + l\rho - \frac{n}{2} > 0$,

$$\leq C\sum_{N=1}^{\infty}(2^N)^{-l}d^{-m+l\rho-\frac{n}{2}}(2^N d)^{\frac{n}{2}}M_2f_2(x_0)$$
$$\leq C\sum_{N=1}^{\infty} d^{-m+l\rho-l}(2^N)^{\frac{n}{2}-l}M_2f_2(x_0).$$

The choice of l assures us that $-m + l\rho - l \geq 0$ and $\frac{n}{2} - l < 0$. Thus

$$|Af_2(x) - C_I| \leq CM_2f_2(x_0).$$

References

1. R. Beals, *A general calculus of pseudodifferential operators*, Duke Math. J. **42** (1975), 1–42.
2. S. Chanillo and A. Torchinsky, *Sharp function and weighted L^p estimates for class of pseudodifferential operators*, Ark. Math. **24** (1986), 1–25.
3. L. Hörmanderpaper Pseudodifferential operators of type (1.1), Comm. Partial Diff. Eq. **13:9** (1988).
4. J-L. Journé, *A boundedness criterian for generalized Calderón-Zygmund operators*, Annl. Math. **120** (1984), 371–397.
5. N. Miller, *Weighted sobolev spaces and pseudodifferential operators with smooth symbols*, Trans. Math. Soc. **269** (1982), 91–109.
6. L. Rodino, *On the boundedness of pseudodifferential operators in the class $L^m_{\rho,1}$*, Proc. Amer. Math. Soc. **58** (1976), 211–215.
7. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, New Jersey, 1971

Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea