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A COMMOM FIXED POINT THEOREM AND ITS APPLICATION

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1. Introduction

The concept of 2-metric spaces has been investigated initially by S. Gähler in a series of papers [2]–[4] and has been developed extensively by himself and many others.

A *2-metric space* (X, d) is a set X with a real-valued function d on $X \times X \times X$ satisfying the following conditions:

(M_1) for two distinct points x and y in X, there exists a point z in X such that $d(x, y, z) \neq 0$,
(M_2) $d(x, y, z) = 0$ if at least two of x, y and z are equal,
(M_3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$,
(M_4) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all x, y, z and w in X.

It has been shown by S. Gähler [2] that a 2-metric d is non-negative and although d is a continuous function of any one of its three arguments, it need not be continuous in two arguments. If it is continuous in two arguments, then it is continuous in all three arguments. A 2-metric d which is continuous in all of its arguments will be said to be *continuous*.

On the other hand, a number of mathematicans have studied the aspects of fixed point theory in the setting of the 2-metric spaces. They have been motivated by various concepts already known for ordinary metric spaces and have thus introduced analogues of various concepts in the frame work of the 2-metric spaces.

Especially, S. V. R. Naidu and J. R. Prasad [9] introduced the concept of weakly commuting mappings as a generalization of commuting mappings. Using this concept, A. Constantin [1] gave a common fixed point theorem in 2-metric spaces. This result generalized, improved and unified some of the results of K. Iséki et al. [5], M. S. Khan and B. Fisher [7], T. Kubiak [8] and S. L. Singh et al. [10]. Recently, in [6], M.

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S. Khan and Y. J. Cho extended the concept of compatible mappings in metric spaces, which is more general than the concept of commuting and weakly commuting mappings, to the setting of 2-metric spaces.

In this paper, we extend a result of A. Constantin [1] by employing compatible mappings instead of weakly commuting mappings under given conditions which is weaker than the condition of [1] and give an application.

2. Preliminaries

The following definitions and lemmas are given in [5]–[7], [9] and [10]:

DEFINITION 2.1. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be *convergent* to a point x in X if $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all a in X. Then x is called the *limit* of the sequence $\{x_n\}$ in X.

DEFINITION 2.2. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a *Cauchy sequence* if $\lim_{m,n\to\infty} d(x_m, x_n, a) = 0$ for all a in X.

DEFINITION 2.3. A 2-metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

Note that in a 2-metric space a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X [9].

DEFINITION 2.4. A mapping S from a 2-metric space (X, d) into itself is said to be *sequentially continuous* at a point x in X if for every sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all a in X, $\lim_{n\to\infty} d(Sx_n, Sx, a) = 0$.

DEFINITION 2.5. Let A and B be two mappings from a 2-metric space (X, d) into itself. Then A and B are said to be *weakly commuting mappings* on X if $d(STx, TSx, a) \leq d(Tx, Sx, a)$ for all x and a in X.

Obviously, any weakly commuting mappings are commuting, but the converse is not necessarily true [9].

DEFINITION 2.6. Let A and B be two mappings from a 2-metric space (X, d) into itself. Then A and B are said to be *compatible mappings* on X if $\lim_{n\to\infty} d(ABx_n, BAx_n, a) = 0$ for all a in X, whenever

$\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some point t in X.

Clearly, any weakly commuting mappings are compatible, but the converse is not necessarily true.

LEMMA 2.1. *Let A and B be compatible mappings from a 2-metric space (X, d) into itself. Suppose that $At = Bt$ for some t in X. Then $ABt = BAt$.*

LEMMA 2.2. *Let A and B be compatible mappings from a 2-metric space (X, d) into itself. Suppose that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some point t in X. Then $\lim_{n\to\infty} BAx_n = At$ if A is sequentially continuous.*

3. A fixed point theorem

Throughout this paper, let (X, d) be a 2-metric space with the continuous 2-metric d. Let $\mathbf{N}$ and $\mathbf{R}^+$ be the sets of all natural numbers and non-negative real numbers, respectively, and let Φ and Φ^* denote the families of all upper-semicontinuous functions $\phi : (\mathbf{R}^+)^5 \to \mathbf{R}^+$ with the following conditions (C_1), (C_2) and (C_4), and (C_1), (C_3) and (C_5), respectively:

(C_1) ϕ is non-decreasing in the 4^{th} and 5^{th} variables,

(C_2) let $v, w \in \mathbf{R}^+$ be such that $v \le \phi(w, w, v, v + w, 0)$ or $v \le \phi(w, v, w, v + w, 0)$ or $v \le \phi(w, w, v, 0, v + w)$ or $v \le \phi(w, v, w, 0, v + w)$. Then $v \le h\,w$, for some $h \in (0, 1)$,

(C_3) let $v, w \in \mathbf{R}^+$ be such that $v \le \phi(w, v, w, v + w, 0)$ or $v \le \phi(w, w, v, 0, v + w)$. Then $v \le h\,w$, for some $h \in (0, 1)$,

(C_4) let $v \in \mathbf{R}^+$ be such that $v \le \phi(v, 0, 0, v, v)$ or $v \le \phi(0, v, 0, v, v)$ or $v \le \phi(0, 0, v, v, v)$. Then $v = 0$,

(C_5) let $v \in \mathbf{R}^+$ be such that $v \le \phi(v, 0, 0, v, v)$. Then $v = 0$.

REMARK 3.1. In [1], the family Φ is introduced by A. Constantin, and Φ^* is a subclass of Φ.

Let A, B, S and T be mappings from a 2-metric space (X, d) into itself satisfying the following conditions:

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X), \tag{3.1}$$

$$d(Ax, By, a) \le \phi\big(d(Sx, Ty, a), d(Ax, Sx, a), d(By, Ty, a), d(Ax, Ty, a), d(By, Sx, a)\big) \tag{3.2}$$

for all x, y and a in X, where $\phi \in \Phi^*$. Then for an arbitrary point x_0 in X, by (3.1), we choose a point x_1 in X such that $Tx_1 = Ax_0$ and for this point x_1, there exists a point x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that

$$(3.3) \qquad y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n} = Bx_{2n+1}$$

for every $n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

LEMMA 3.1. *Let A, B, S and T be mappings from a 2-metric space (X, d) into itself satisfying the conditions (3.1) and (3.2). Then the sequence $\{y_n\}$ defined by (3.3) is a Cauchy sequence in X.*

Proof. In (3.2), taking $x = x_{2n+2}$, $y = x_{2n+1}$ and $a = y_{2n}$, we have

$$\begin{aligned} d(y_{2n+2}, y_{2n+1}, y_{2n}) &= d(Ax_{2n+2}, Bx_{2n+1}, y_{2n}) \\ &\le \phi\big(d(y_{2n+1}, y_{2n}, y_{2n}), d(y_{2n+2}, y_{2n+1}, y_{2n}), \\ &\qquad d(y_{2n+1}, y_{2n}, y_{2n}), d(y_{2n+2}, y_{2n}, y_{2n}), \\ &\qquad d(y_{2n+1}, y_{2n+1}, y_{2n})\big) \end{aligned}$$

and so $d(y_{2n+2}, y_{2n+1}, y_{2n}) = 0$ by (C_1) and (C_3). Similarly, we have $d(y_{2n+1}, y_{2n+2}, y_{2n+3}) = 0$. Hence, we obtain

$$(3.4) \qquad d(y_n, y_{n+1}, y_{n+2}) = 0$$

for every $n \in \mathbf{N}_0$. For all a in X, we denote $d_n(a) = d(y_n, y_{n+1}, a)$, $n = 0, 1, 2, \cdots$. By (3.4), we have

$$\begin{aligned} d(y_n, y_{n+2}, a) &\le d(y_n, y_{n+2}, y_{n+1}) + d(y_n, y_{n+1}, a) + d(y_{n+1}, y_{n+2}, a) \\ &= d_n(a) + d_{n+1}(a). \end{aligned}$$

Taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (3.2), we have

$$\begin{aligned} d_{2n+1}(a) &= d(y_{2n+2}, y_{2n+1}, a) \\ &= d(Ax_{2n+2}, Bx_{2n+1}, a) \\ &\le \phi\big(d(y_{2n+1}, y_{2n}, a), d(y_{2n+2}, y_{2n+1}, a), \\ &\qquad d(y_{2n+1}, y_{2n}, a), d(y_{2n+2}, y_{2n}, a), d(y_{2n+1}, y_{2n+1}, a)\big) \\ &= \phi\big(d_{2n}(a), d_{2n+1}(a), d_{2n}(a), d_{2n}(a) + d_{2n+1}(a), 0\big) \end{aligned}$$

for all a in X. By (C_3), we obtain

$$d_{2n+1}(a) \leq h\, d_{2n}(a).$$

Similarly, we have

$$d_{2n+2}(a) \leq h\, d_{2n+1}(a).$$

In general, we obtain

$$d_n(a) \leq h\, d_{n-1}(a) \leq h^2 d_{n-2}(a) \leq \cdots\cdots \leq h^n d_0(a) \tag{3.5}$$

for all a in X, where $0 < h < 1$.

By using the these facts, we have the following:

A. $d_0(y_0) = 0 \Longrightarrow d_n(y_0) = 0$ for every $n \in \mathbf{N}$.

B. $d_{m-1}(y_m) = 0$ for any $m \in \mathbf{N} \Longrightarrow d_n(y_m) = 0$ for all $n \geq m-1$.

C. $d_{m-1}(y_{n+1}) = 0 = d_{m-1}(y_n)$ for $0 \leq n < m-1$ and (M_4)
$\Longrightarrow d_n(y_m) \leq d_n(y_{m-1}) \leq d_n(y_{m-2}) \leq \cdots\cdots$.

D. $d_n(y_{n+1}) = 0 \Longrightarrow d_n(y_m) = 0$ for $0 \leq n < m-1$.

Thus, we have shown $d_n(y_m) = 0$ for all $m, n \in \mathbf{N}_0$.

E. $d_{j-1}(y_i) = 0 = d_{j-1}(y_k)$ for any $i, j, k \in \mathbf{N}_0$ with $i < j$
$\Longrightarrow d(y_i, y_j, y_k) \leq d(y_i, y_{j-1}, y_k)$.

Therefore, by using the above inequality in **E**, repeatedly, we have

$$d(y_i, y_j, y_k) \leq d(y_i, y_i, y_k) = 0,$$

which means that, for every $i, j, k \in \mathbf{N}_0$,

$$d(y_i, y_j, y_k) = 0. \tag{3.6}$$

If $m \geq n$, then, by (3.5) and (3.6), we have

$$\begin{aligned}
d(y_n, y_m, a) &\leq d(y_n, y_m, y_{n+1}) + d(y_n, y_{n+1}, a) + d(y_{n+1}, y_m, a)\\
&= d(y_n, y_{n+1}, a) + d(y_{n+1}, y_m, a)\\
&\leq d(y_n, y_{n+1}, a) + d(y_{n+1}, y_{n+2}, a) + d(y_{n+2}, y_m, a)\\
&\leq \cdots\\
&\leq d(y_n, y_{n+1}, a) + d(y_{n+1}, y_{n+2}, a) + \cdots + d(y_{m-1}, y_m, a)\\
&\leq \{h^n + h^{n+1} + \cdots + h^{m-1}\} d(y_0, y_1, a)\\
&\leq \frac{h^n}{1-h} d(y_0, y_1, a)
\end{aligned}$$

for all a in X. Therefore, since $0 < h < 1$, $\{y_n\}$ is a Cauchy sequence in X. This completes the proof.

THEOREM 3.2. *Let A, B, S and T be mappings from a complete 2-metric space (X, d) into itself satisfying the conditions* (3.1) *and* (3.2). *Suppose that*
(3.7) *one of A, B, S and T is sequentially continuous and*
(3.8) *the pairs A, S and B, T are compatible.*
Then A, B, S and T have a unique common fixed point in X.

Proof. Let $\{y_n\}$ be the sequence in X defined by (3.3). By Lemma 3.1, $\{y_n\}$ is a Cauchy sequence and hence it converges to some point z in X. Consequently, the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to z.

Now, suppose that S is sequentially continuous. Since A and S are compatible, Lemma 2.2 implies

$$S^2x_{2n} \text{ and } ASx_{2n} \to Sz \quad \text{as } n \to \infty.$$

By (3.2), we obtain

$$\begin{aligned} d(ASx_{2n}, Bx_{2n+1}, a) \leq & \phi\big(d(S^2x_{2n}, Tx_{2n+1}, a), d(ASx_{2n}, S^2x_{2n}, a), \\ & d(Bx_{2n+1}, Tx_{2n+1}, a), d(ASx_{2n}, Tx_{2n+1}, a), \\ & d(Bx_{2n+1}, S^2x_{2n}, a)\big) \end{aligned}$$

for all a in X. Letting $n \to \infty$, we have

$$d(Sz, z, a) \leq \phi\big(d(Sz, z, a), 0, 0, d(Sz, z, a), d(z, Sz, a)\big),$$

so that $z = Sz$ by (C_5). By (3.2), we also obtain

$$\begin{aligned} d(Az, Bx_{2n+1}, a) \leq \phi\big(& d(Sz, Tx_{2n+1}, a), d(Az, Sz, a), \\ & d(Bx_{2n+1}, Tx_{2n+1}, a), d(Az, Tx_{2n+1}, a), \\ & d(Bx_{2n+1}, Sz, a)\big) \end{aligned}$$

for all a in X. Letting $n \to \infty$, we have

$$d(Az, z, a) \leq \phi\big(d(Sz, z, a), d(Az, Sz, a), 0, d(Az, z, a), d(z, Sz, a)\big),$$

so that $z = Az$ by (C_3). Since $A(X) \subset T(X)$, $z \in T(X)$ and hence there exists a point u in X such that $z = Az = Tu$. For all a in X,

$$\begin{aligned} d(z, Bu, a) &= d(Az, Bu, a) \\ &\leq \phi\big(d(Sz, Tu, a), 0, d(Bu, Tu, a), \\ &\qquad d(Az, Tu, a), d(Bu, z, a)\big), \end{aligned}$$

which implies that $z = Bu$. Since B and T are compatible and $Tu = Bu = z$, $Tz = TBu = BTu = Bz$ by Lemma 2.1. Moreover, by (3.2), we obtain, for all a in X,

$$\begin{aligned} d(z, Tz, a) &= d(Az, Bz, a) \\ &\le \phi\big(d(z, Tz, a), 0, d(Bz, Tz, a), d(z, Tz, a), d(Bz, z, a)\big), \end{aligned}$$

so that $z = Tz$ by (C_5). Therefore, the point z is a common fixed point of A, B, S and T. Similarly, we can also complete the proof when T is sequentially continuous.

Next, suppose that A is sequentially continuous. Since A and S are compatible, it follows from Lemma 2.2 that

$$A^2x_{2n} \text{ and } SAx_{2n} \to Az \quad \text{as } n \to \infty.$$

By (3.2), we have

$$\begin{aligned} d(A^2x_{2n}, Bx_{2n+1}, a) \le \phi\big(&d(SAx_{2n}, Tx_{2n+1}, a), d(A^2x_{2n}, SAx_{2n}, a), \\ &d(Bx_{2n+1}, Tx_{2n+1}, a), d(A^2x_{2n}, Tx_{2n+1}, a), \\ &d(Bx_{2n+1}, SAx_{2n}, a)\big) \end{aligned}$$

for all a in X. Letting $n \to \infty$, we obtain

$$d(Az, z, a) \le \phi\big(d(Az, z, a), 0, 0, d(Az, z, a), d(z, Az, a)\big),$$

so that $z = Az$. Hence, there exists a point v in X such that $z = Az = Tv$. Thus we have

$$\begin{aligned} d(A^2x_{2n}, Bv, a) \le \phi\big(&d(SAx_{2n}, Tv, a), d(A^2x_{2n}, SAx_{2n}, a), \\ &d(Bv, Tv, a), d(A^2x_{2n}, Tv, a), d(Bv, SAx_{2n}, a)\big) \end{aligned}$$

for all a in X. Letting $n \to \infty$, it follows that

$$d(z, Bv, a) \le \phi\big(d(z, Tv, a), 0, d(Bv, Tv, a), d(Az, Tv, a), d(Bv, z, a)\big),$$

which implies that $z = Bv$. Since B and T are compatible and $Tv = Bv = z$, $Tz = TBv = BTv = Bz$. Moreover, by (3.2), we have

$$\begin{aligned} d(Ax_{2n}, Bz, a) \le \phi\big(&d(Sx_{2n}, Tz, a), d(Ax_{2n}, Sx_{2n}, a), \\ &d(Bz, Tz, a), d(Ax_{2n}, Tz, a), d(Bz, Sx_{2n}, a)\big) \end{aligned}$$

for all a in X. Letting $n \to \infty$, it follows that

$$d(z, Bz, a) \leq \phi\big(d(z, Tz, a), 0, d(Bz, Tz, a), d(z, Tz, a), d(Bz, z, a)\big),$$

so that $z = Bz$. Since $B(X) \subset S(X)$, there exists a point w in X such that $z = Bz = Sw$. Thus, for all a in X, we have

$$\begin{aligned} d(Aw, z, a) &= d(Aw, Bz, a) \\ &\leq \phi\big(d(Sw, z, a), d(Aw, Sw, a), 0, d(Aw, z, a), d(z, Sw, a)\big), \end{aligned}$$

so that $Aw = z$. Since A and S are compatible and $Aw = Sw = z$, $Sz = SAw = ASw = Az$. Therefore, the point z is a common fixed point of A, B, S and T. Similarly, we can also complete the proof when B is sequentially continuous.

It follows easily from (3.2) that z is a unique common fixed point of A, B, S and T. This completes the proof.

REMARK 3.2. It is not hard to verify that a part of theorem in [1] holds under given conditions in Theorem 3.2.

REMARK 3.3. Theorem 3.2 extends a result of A. Constantin [1] by assuming compatibility, $\phi \in \Phi^*$ and any one of sequentially continuous mappings in stead of weak commutativity, $\phi \in \Phi$ and two of sequentially continuous mappings, respectively.

REMARK 3.4. Our result also extends and improves some results of K. Iséki et al. [5], M. S. Khan and B. Fisher [7], T. Kubiak [8], S. V. R. Naidu and J. R. Prasad [9] and S. L. Singh et al. [10].

4. Applications

By applying Theorem 3.2, we can show the existence of solutions for a equation of the form

$$Ax = Px,$$

where A and P are sequentially continuous mappings from a complete 2-metric space (X, d) into itself.

THEOREM 4.1. *Let A and P are sequentially continuous mappings from a complete 2-metric space (X, d) into itself satisfying the following conditions: there exist $0 < \beta \leq \alpha$ and $m \in \mathbb{N}$ such that*

$$d(Ax, Ay, a) \geq \alpha\, d(x, y, a), \tag{4.1}$$

$$P^m(X) \subset AP^{m-1}(X), \tag{4.2}$$

$$\begin{aligned} d(P^m x, P^m y, a) \leq \beta\, \phi\big(&d(P^{m-1}x, P^{m-1}y, a), \\ &d(A^{-1}P^m x, P^{m-1}x, a), \\ &d(A^{-1}P^m y, P^{m-1}y, a), \\ &d(A^{-1}P^m x, P^{m-1}y, a), \\ &d(A^{-1}P^m y, P^{m-1}x, a)\big) \end{aligned} \tag{4.3}$$

for all x, y and a in X, where $\phi \in \Phi^*$. Suppose that
(4.4) *A is surjective and*
(4.5) ~~*the pair*~~ *$A^{-1}P^m, P^{m-1}$* ~~*is compatible*~~*.*
Then the equation $Ax = Px$ has at least one solution in X.

Proof. We note that if $Ax = Ay$, then $x = y$, so that A is bijective and hence A^{-1} exists. From (4.1) and (4.3), we deduce

$$\begin{aligned} d(A^{-1}P^m x, A^{-1}P^m y, a) &\leq \frac{1}{\alpha}\, d(P^m x, P^m y, a) \\ &\leq \frac{\beta}{\alpha}\, \phi\big(d(P^{m-1}x, P^{m-1}y, a), \\ &\qquad d(A^{-1}P^m x, P^{m-1}x, a), \\ &\qquad d(A^{-1}P^m y, P^{m-1}y, a), \\ &\qquad d(A^{-1}P^m x, P^{m-1}y, a), \\ &\qquad d(A^{-1}P^m y, P^{m-1}x, a)\big) \end{aligned}$$

for all a in X. Now, we see that all the hypotheses of Theorem 3.2 for $A^{-1}P^m$ and P^{m-1} are satisfied. Therefore, there exists a unique point x_0 in X such that

$$A^{-1}P^m x_0 = P^{m-1}x_0 = x_0$$

and so we can deduce $A^{-1}Px_0 = x_0$, that is $Ax_0 = Px_0$. This completes the proof.

COROLLARY 4.2. *Let A and P be sequentially continuous mappings from a complete 2-metric space (X, d) into itself satisfying the conditions* (4.1) *and* (4.4). *Suppose that there exist $0 < \beta \leq \alpha$ and $0 < h < 1$ such that*

$$d(Px, Py, a) \leq \beta h\, d(x, y, a) \tag{4.6}$$

for all x, y and a in X. Then the equation $Ax = Px$ has at least one solution in X.

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