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# AN EQUIVALENT CONDITION FOR MEMBERSHIP IN NEW CLASSES $\mathbf{A}^{l}_{m,n}$

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## 1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert spaces and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that cointains $T$ and $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology. Moreover, let $Q_{\mathcal{A}_T}$ denote the quotient space $\mathcal{C}_1(\mathcal{H})/\perp_{\mathcal{A}_T}$, where $\mathcal{C}_1(\mathcal{H})$ is the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and $\perp_{\mathcal{A}_T}$ denotes the preannihilator of $\mathcal{A}_T$ in $\mathcal{C}_1(\mathcal{H})$. For a brief notation, we shall denote $Q_{\mathcal{A}_T}$ by $Q_T$. One knows that $\mathcal{A}_T$ is the dual space of $Q_T$ and that the duality is given by

$$\langle A, [L]\rangle = tr(AL), \quad A \in \mathcal{A}_T, [L] \in Q_T. \tag{1}$$

The Banach space $Q_T$ is called a predual of $\mathcal{A}_T$. For $x$ and $y$ in $\mathcal{H}$, we can write $x \otimes y$ for the rank one operator in $\mathcal{C}_1(\mathcal{H})$ defined by

$$(x \otimes y)(u) = (u, y)x \quad \text{for} \quad \text{all} \quad u \in \mathcal{H}. \tag{2}$$

The theory of dual algebras is applied to the study of invariant subspaces, dilation theory, and reflexivity. The classes $\mathcal{A}_{m,n}$ (to be defined in section 2) were defined by Bercovici-Foias-Pearcy in [2]. Also these classes are closely related to the study of the theory of dual algebras. C. Apostol, H. Bercovici, C. Foias and C. Pearcy [1] studied geometric criteria for membership in the class $A_{\aleph_0} = A_{\aleph_0,\aleph_0}$ (to be defined in section 2) S. Brown, B. Chevreau, G. Exner and C. Pearcy [5], [7],

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[8] obtained topological criteria and geometric criteria for membership in the class $A_{\aleph_0}$ or $A_{1,\aleph_0}$.In this paper we construct new classes and obtain an equivalent condition for membership in the new classes.

## 2. Notation and preliminaries

The notation and terminology employed herein agree with those in [3], [4], [11]. We shall denote by $D$ the open unit disc in the complex plane $C$, and we write $\mathbf{T}$ for the boundary of $D$. The space $L^p = L^p(\mathbf{T}), 1 \leq p \leq \infty$, is the usual Lebesgue function space relative to normalized Lebesgue measure $m$ on $\mathbf{T}$. The space $H^p = H^p(\mathbf{T}), 1 \leq p \leq \infty$, is the usual Hardy space. It is well-known that the space $H^\infty$ is the dual space of $L^1/H_0^1$, where

$$(3) \qquad H_0^1 = \{f \in L^1 : \int_0^{2\pi} f(e^{it})e^{int}dt = 0, \quad \text{for} \quad n = 0, 1, 2, \cdots\}$$

and the duality is given by the pairing

$$(4) \qquad \langle f, [g]\rangle = \int_T fg\ dm \quad \text{for} \quad f \in H^\infty, [g] \in L^1/H_0^1.$$

Recall that any contraction $T$ can be written as a direct sum $T = T_1 \oplus T_2$, where $T_1$ is a completely nonunitary contraction and $T_2$ is a unitary operator. If $T_2$ is absolutely continuous or acts on the space $(0)$, $T$ will be called an *absolutely continuous contraction.* The following Foias–Sz. Nagy functional calculus [3, Theorem 4.1] provides a good relationship between the function space $H^\infty$ and a dual algebra $\mathcal{A}_T$.

THEOREM 2.1. ([3. Theorem 4.1]) *Let $T$ be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there is an algebra homomorphism $\Phi_T : H^\infty \to \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ such that*
*(a) $\Phi_T(1) = 1_{\mathcal{H}}, \quad \Phi_T(\xi) = T$,*
*(b) $\|\Phi_T(f)\| \leq \|f\|_\infty, \quad f \in H^\infty$,*
*(c) $\Phi_T$ is continuous if both $H^\infty$ and $\mathcal{A}_T$ are given their weak* topologies,*
*(d) the range of $\Phi_T$ is weak* dense in $\mathcal{A}_T$,*
*(e) there exists a bounded, linear, one-to-one map $\phi_T : Q_T \to L^1/H_0^1$ such that ${\phi_T}^* = \Phi_T$, and*
*(f) if $\Phi_T$ is an isometry, then $\Phi_T$ is a weak* homeomorphism of $H^\infty$ onto $\mathcal{A}_T$ and $\phi_T$ is an isometry of $Q_T$ onto $L^1/H_0^1$.*

DEFINITION 2.2. ([9]) Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra and let $m$ and $n$ be any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbf{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \quad 0 \leq i < m, 0 \leq j < n, \tag{5}$$

where $\{[L_{ij}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from $Q_{\mathcal{A}}$, has a solution $\{x_i\}_{0 \leq i < m}$, $\{y_j\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$.

For brief notation, we shall denote $(\mathbf{A}_{n,n})$ by $(\mathbf{A}_n)$. We denote by $\mathbf{A} = \mathbf{A}(\mathcal{H})$ the class of all absolutely continuous contractions $T$ in $\mathcal{L}(\mathcal{H})$ for which the Foias-Sz.Nagy functional calculus $\Phi_T : \mathcal{H}^\infty \to \mathcal{A}_T$ is an isometry. Furthermore, if $m$ and $n$ are cardinal numbers such that $1 \leq m, n \leq \aleph_0$, we denote by $\mathbf{A}_{m,n} = \mathbf{A}_{m,n}(\mathcal{H})$ the set of all $T$ in $\mathbf{A}(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(\mathbf{A}_{m,n})$.

To establish our results, it will be convenient to use the minimal coisometric extension theorem [11]: every contraction $T$ in $\mathcal{L}(\mathcal{H})$ has a minimal coisometric extension $B = B_T$ that is unique up to unitary equivalence.

Given such $T$ and $B$, one knows that there exists a canonical decomposition of the isometry $B^*$ as

$$B^* = S \oplus R^* \tag{6}$$

corresponding to a decomposition of the space

$$\mathcal{K} = \mathcal{S} \oplus \mathcal{R}, \tag{7}$$

where, if $\mathcal{S} \neq (0)$, $S$ is a unilateral shift operator of some multiplicity in $\mathcal{L}(\mathcal{S})$, and, if $\mathcal{R} \neq (0)$, $R$ is a unitary operator in $\mathcal{L}(\mathcal{R})$. Of course, either $\mathcal{S}$ or $\mathcal{R}$ may be $(0)$. ([5])

Let $P_\lambda$ be the Poisson kernel function

$$P_\lambda(e^{it}) = (1 - |\lambda|^2)|1 - \bar{\lambda}e^{it}|^{-2}, \quad e^{it} \in \mathbf{T}, \tag{8}$$

in $L^1$, for each $\lambda \in D$. Then it follows from [3,p34] that

$$(9) \qquad \langle f, [P_\lambda] \rangle = \tilde{f}(\lambda), \quad f \in H^\infty,$$

where $\tilde{f}$ is the analytic extension of $f$ to $D$. For a given contraction $T \in \mathbf{A}(\mathcal{H})$, let us denote $\phi_T^{-1}([P_\lambda]) = [C_\lambda]$. Then we have

$$(10) \qquad \langle f(T), [C_\lambda] \rangle = \tilde{f}(\lambda), \quad f \in H^\infty.$$

LEMMA 2.3. ([5, Lemma 3.5]) *Suppose $T \in \mathbf{A}(\mathcal{H})$ and has minimal coisometric extension $B$ in $\mathcal{L}(\mathcal{K})$. Then $B \in \mathbf{A}(\mathcal{K}), \Phi_T \circ \Phi_B^{-1}$ is an isometry and weak* homeomorphism from $\mathcal{A}_B$ onto $\mathcal{A}_T$,and $\jmath = \varphi_B^{-1} \circ \varphi_T$ is a linear isometry of $Q_T$ onto $Q_B$. Moreover,*

$$(11) \qquad j([C_\lambda]_T) = [C_\lambda]_B, \quad \lambda \in D$$

and

$$(12) \qquad j([x \otimes y]_T) = [x \otimes y]_B, \quad x, y \in \mathcal{H}.$$

LEMMA 2.4. ([5, Lemma 3.6]) *If $T$ belongs to $\mathbf{A}(\mathcal{H})$ and has minimal coisometric extension $B$ in $\mathcal{L}(\mathcal{K})$, $x, y \in \mathcal{H}$, and $w, z \in \mathcal{K}$, then*

$$(13) \qquad \|[x \otimes y]_T\| = \|[x \otimes y]_B\|,$$

$$(14) \qquad [x \otimes z]_B = [x \otimes Pz]_B,$$

*and*

$$(15) \qquad [w \otimes z]_B = [Qw \otimes Qz]_B + [Aw \otimes Az]_B.$$

LEMMA 2.5. ([5, Proposition 4.5]) *Suppose $T \in \mathbf{A}(\mathcal{H})$ and has minimal coisometric extension $B$ in $\mathcal{L}(\mathcal{S} \oplus \mathcal{R})$ and suppose that for every $[L]$ in $Q_T$ there exists a Cauchy sequence $\{x_n\}$ in $\mathcal{H}$ and sequences $\{w_n\}$ in $\mathcal{S}$ and $\{b_n\}$ in $\mathcal{R}$ such that $\{w_n + b_n\}$ is bounded and $\|(\varphi_B^{-1} \circ \varphi_T)([L]_T) - [x_n \otimes (w_n + b_n)]_B\| \to 0$. Then $T \in \mathbf{A}_1$.*

We shall employ the notation $C_{\cdot 0} = C_{\cdot 0}(\mathcal{H})$ for the class of all (completely nonunitary) contractions $T$ in $\mathcal{L}(\mathcal{H})$ such that the sequences $\{T^*\}^n$ converges to zero in the strong operator topology and is denoted by,as usual,$C_{0\cdot} = (C_{\cdot 0})^*$,and $\mathbf{N}$ is denoted by the set of all natural numbers.

LEMMA 2.6. ([6, Theorem 2.1]) *Suppose* $\{T_k\}_{k=1}^{\infty}$ *is any sequence of operators contained in the class* $\mathbf{A}_{\aleph_0} \cap C_{0}$, $\{[L_k]_{T_k}\}_{k=1}^{\infty}$ *is an arbitrary sequence (where* $[L_k]_{T_k} \in Q_{T_k}$*), and* $\{\epsilon_k\}_{k=1}^{\infty}$ *is any sequence of positive numbers. Then there exists a dense set* $\mathcal{D} \subset \mathcal{H}$ *such that for every* $x$ *in* $\mathcal{D}$*, there exists a sequence* $\{y_k^x\}_{k=1}^{\infty} \subset \mathcal{H}$ *satisfying*

$$[x \otimes y_k^x]_{T_k} = [L_k]_{T_k}, \quad k \in \mathbf{N}, \tag{16}$$

*and*

$$\|y_k^x\| > \epsilon_k, \quad k \in \mathbf{N}. \tag{17}$$

## 3. Classes $\mathcal{A}^l_{m,n}(\mathcal{H})$ and an equivalent condition for membership in $\mathcal{A}^l_{m,n}(\mathcal{H})$

From the idea of lemma 2.6, we construct new classes as following :

DEFINITION 3.1. Let $m, n$ and $l$ be any cardinal numbers such that $1 \le m, n, l \le \aleph_0$. We denote by $\mathbf{A}^l_{m,n}(\mathcal{H})$ the class of all $\{T_k\}_{k=1}^{l}$ in $\mathbf{A}(\mathcal{H})$ such that every $m \times n \times l$ system of simultaneous equations of the form

$$[x_i \otimes y_j^{(k)}]_{T_k} = [L_{ij}^{(k)}]_{T_k}, \tag{18}$$

where $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0 \le i < m \\ 0 \le j < n}}$ is an arbitrary $m \times n$ array from $Q_{T_k}$ for each $1 \le k \le l$, has a solution $\{x_i\}_{0 \le i < m}, \{y_j^{(k)}\}_{\substack{0 \le j < n \\ 1 \le k \le l}}$ consisting of a pair of sequences of vectors from $\mathcal{H}$.

REMARK 3.2. If $\{T_k\}_{k=1}^{\infty}$ are in the class $\mathbf{A}_{\aleph_0} \cap C_{0}$, then $\{T_k\}_{k=1}^{\infty} \in \mathbf{A}^{\aleph_0}_{1,1}$, by lemma 2.6.

We are now ready to prove our main theorem.

THEOREM 3.3. *Suppose* $m, n$ *and* $l$ *are cardinal numbers such that* $1 \le m, n, l \le \aleph_0$ *and* $T_k \in \mathbf{A}(\mathcal{H})$ *has minimal coisometric extension* $B_k$ *in* $\mathcal{L}(\mathcal{S}_k \oplus \mathcal{R}_k)$ *for* $k$, $1 \le k \le l$*. Then* $\{T_k\}_{k=1}^{l} \in \mathbf{A}^l_{m,n}$ *if and only if for* $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0 \le i < m \\ 0 \le j < n}} \subset Q_{T_k}$ *for* $k$, $1 \le k \le l$*, there exists a Cauchy* *sequence* $\{x_{i,p}\}_{p=1}^{\infty}$ *in* $\mathcal{H}$ *and sequences* $\{w_{j,p}^{(k)}\}_{p=1}^{\infty}$ *in* $\mathcal{S}_k$ *and* $\{b_{j,p}^{(k)}\}_{p=1}^{\infty}$ *in* $\mathcal{R}_k$ *such that* $\{w_{j,p}^{(k)}+b_{j,p}^{(k)}\}$ *is bounded and* $\|(\varphi_{B_k}^{-1}\circ\varphi_{T_k})([L_{ij}^{(k)}]_{T_k})-[x_{i,p}\otimes(w_{j,p}^{(k)}+b_{j,p}^{(k)})]_{B_k}\|\to 0$.

*Proof.* The idea of this proof comes from Lemma 2.5. Suppose $\{T_k\}_{k=1}^{l}\in\mathbf{A}_{m,n}^{l}(\mathcal{H})$. It follows from the definition of $\mathbf{A}_{m,n}^{l}(\mathcal{H})$ that, for $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0\le i<m\\0\le j<n}}$ in $Q_{T_k}$ for $k$, $1\le k\le l$, there exist $x_i, y_j^{(k)}\in\mathcal{H}, 0\le i<m, 0\le j<n, 1\le k\le l$ such that $[L_{ij}^{(k)}]_{T_k}=[x_i\otimes y_j^{(k)}]_{T_k}$. Set $x_{i,p}=x_i, y_{j,p}^{(k)}=y_j^{(k)}=w_j^{(k)}+b_j^{(k)}\in\mathcal{S}_k\oplus\mathcal{R}_k$ for any $p\in\mathbf{N}$. Then it is obvious that these are required sequences.
Conversely, let us $v_{j,p}^{(k)}=\mathbf{P}(w_{j,p}^{(k)}+b_{j,p}^{(k)}), p\in\mathbf{N}$, where $\mathbf{P}$ is an orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. Since $\{v_{j,p}^{(k)}\}_{p=1}^{\infty}$ is bounded, we may suppose w.l.o.g, that $\{v_{j,p}^{(k)}\}_{p=1}^{\infty}$ converges weakly to $v_j^k$. Moreover, since $\{x_{i,p}\}_{p=1}^{\infty}$ is a Cauchy sequence, we have $\{x_{i,p}\}$ converges strongly to $x_i$.

$$\begin{aligned}&\|[x_i\otimes v_{j,k}^{(k)}]_{T_k}-[x_{i,p}\otimes v_{j,p}^{(k)}]_{T_k}\|\\&=\|[(x_i-x_{i,p})\otimes v_{j,p}^{(k)}]_{T_k}\|\\&\le\|x_i-x_{i,p}\|\cdot\|v_{j,p}^{(k)}\|\to 0.\end{aligned}$$

Also from (12) and (14), with $j_k=\varphi_{B_k}^{-1}\circ\varphi_{T_k}$, we have

$$\begin{aligned}&\|[L_{ij}^{(k)}]_{T_k}-[x_{i,p}\otimes v_{j,p}^{(k)}]_{T_k}\|\\&=\|\varphi_{B_k}^{-1}\circ\varphi_{T_k}([L_{ij}^{(k)}]_{T_k})-[x_{i,p}\otimes v_{j,p}^{(k)}]_{B_k}\|\\&=\|\varphi_{B_k}^{-1}\circ\varphi_{T_k}([L_{ij}^{(k)}]_{T_k})-[x_{i,p}\otimes(w_{j,p}^{(k)}+b_{j,p}^{(k)})]_{B_k}\|\to 0.\end{aligned}$$

Then

$$\begin{aligned}&\|[L_{ij}^{(k)}]_{T_k}-[x_i\otimes v_{j,p}^{(k)}]_{T_k}\|\\&\le\|[L_{ij}^{(k)}]_{T_k}-[x_{i,p}\otimes v_{j,p}^{(k)}]_{T_k}\|+\|[x_{i,p}\otimes v_{j,p}^{(k)}]_{T_k}-[x_i\otimes v_{j,p}^{(k)}]_{T_k}\|\to 0.\end{aligned}$$

So

$$\|[L_{ij}^{(k)}]_{T_k}-[x_i\otimes v_{j,p}^{(k)}]_{T_k}\|\to 0$$

We now compute to show that $[L^{(k)}_{ij}]_{T_k} = [x_i \otimes v^{(k)}_j]_{T_k}$, and thus complete the proof ; for $h \in \mathbf{H}^\infty(\mathbf{T})$, we have

$$\langle h(T_k), [L^{(k)}_{ij}]_{T_k}\rangle = \lim_p \langle h(T_k), [x_i \otimes v^{(k)}_{j,p}]_{T_k}\rangle = \lim_p (h(T_k)x_i, v^{(k)}_{j,p})$$
$$= (h(T_k)x_i, v^{(k)}_j) = \langle h(T_k), [x_i \otimes v^{(k)}_j]_{T_k}\rangle.$$

Hence we have $[L^{(k)}_{ij}]_{T_k} = [x_i \otimes v^{(k)}_j]_{T_k}, \quad 0 \le i < m, 0 \le j < n, 1 \le k \le l$.
Therefore the proof is complete.

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