Pusan Kyŏngnam Math J. 8(1992), No 1, pp. 25–39

MAXIMAL SPACELIKE HYPERSURFACES IN A LORENTZIAN MANIFOLD WITH A CONSTANT CURVATURE

SEONG-KOWAN HONG

1. Introduction

A maximal spacelike hypersurface in a Lorentzian manifold is a counterpart of a minimal hypersurface in a Riemannian manifold. Our main purpose here is to study the Bernstein type problem proposed by E. Calabi [3].

In section 2, we study maximal spacelike hypersurfaces in L^3 obtained by revolving spacelike curves about an axis.

In section 3, we give local formulas needed in section 4.

S. Y. Cheng and S. T. Yau proved in [4] that the only maximal spacelike hypersurface which is a closed subset of the Lorentz-Minkowski space is a linear hyperplane. Note that Lorentz-Minkowski space is flat, and all maximal space-like hypersurfaces are totally geodesic. In section 4, we study the Bernstein-type problems proposed by E. Calabi [3] in Lorentzian manifolds with constant curvatures.

2. Rotatory maxiaml spacelike surfaces in L^3

Let us consider a transformation of L^3 which preserves the Lorentz metric, time- and space-orientations. We will call such a transformation a proper rotation in L^3. By a rotatory maximal spacelike surface in L^3 we mean a maximal spacelike surface obtained by properly rotating about an axis a regular spacelike curve lying in some plane containg the axis.

All rotatory maximal spacelike surfaces are characterized by the following theorem.

Received April 26, 1992.

This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1991.

THEOREM 1. *Let M be a connected, nonplanar rotatory maximal spacelike surface. Then M must be parametrized in one of the following ways :*

$$(1)\qquad \begin{cases} \left(\dfrac{\sin(cs+d)}{c}\cosh t, \dfrac{\sin(cs+d)}{c}\sinh t, s\right) \\ \left(s, \dfrac{\sinh(cs+d)}{c}\cos t, \dfrac{\sinh(cs+d)}{c}\sin t\right) \\ \left((c^2s+d)^{1/3}, -\dfrac{t^2}{2}(c^2s+d)^{1/3}+s, t(c^2s+d)^{1/3}\right). \end{cases}$$

Here we use coordinates with respect to the designated frames.

To prove the theorem we need to solve differential equations which arise from the following lemmas.

LEMMA 1. *Let M be a rotatoary spacelike surface in L^3, with rotation axis l.*

(1) If l is spacelike, then M is represented by

$$(2)\qquad \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0(s) \\ 0 \\ x^2(s) \end{bmatrix} = \begin{bmatrix} x^0(s)\cosh t \\ x^0(s)\sinh t \\ x^2(s) \end{bmatrix}$$

with respect to the basis $\{e_0, e_1, e_2\}$, where $l=$ span $\{e_2\}$, $(x^0(s), 0, x^2(s))$ is a regular spacelike curve with $x^0(s)\neq 0$.

(2) If l is timelike, then M is represented by

$$(3)\qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix} \begin{bmatrix} x^0(s) \\ x^1(s) \\ 0 \end{bmatrix} = \begin{bmatrix} x^0(s) \\ x^1(s)\cos t \\ x^1(s)\sin t \end{bmatrix},$$

with respect to the basis $\{e_0, e_1, e_2\}$, where $l=$ span $\{e_0\}$, and $(x^0(s), x^1(s), 0)$ is a regular spacelike curve with $x^1(s)\neq 0$.

(3) If l is lightlike, then M is represented by

$$(4)\qquad \begin{bmatrix} 1 & 0 & 0 \\ -\frac{t^2}{2} & 1 & -t \\ t & 0 & 1 \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \\ 0 \end{bmatrix} = \begin{bmatrix} a(s) \\ -\frac{t^2}{2}a(s)+b(s) \\ ta(s) \end{bmatrix},$$

with respect to the null frame basis $\{A, B, C\}$, *where* $l =$ span $\{B\}$, *and* $(a(s), b(s), 0)$ *is a regular spacelike curve with* $a(s) \neq 0$.

Proof. Let M be generated by a curve $x(s)$ which lies in a plane H containing l.

(1) Suppose l is a spacelike axis. Choose an orthonormal frame $\{e_0, e_1, e_2\}$ of L^3 so that $l =$ span $\{e_2\}$. Then all the proper rotations about l are represented by the matrix

$$\begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad t \in R,$$

with respect to the frame.

Note that H is nondegenerate under the induced metric from L^3, otherwise M would be a degenerate surface. Futhermore, H cannot be spacelike. Suppose H was spacelike in L^3 so that M could be represented by

$$\begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x^1(s) \\ x^2(s) \end{bmatrix} = \begin{bmatrix} x^1(s) \sinh t \\ x^1(s) \cosh t \\ x^2(s) \end{bmatrix},$$

with respect to the basis $\{e_0, e_1, e_2\}$, where $l =$ span $\{e_2\}$, $H =$ span $\{e_1, e_2\}$, and $(0, x^1(s), x^2(s))$ is a regular spacelike curve with $x^1(s) \neq 0$. Then the first fundamental form of M would be

$$\left(\left(\frac{dx^1}{ds} \right)^2 + \left(\frac{dx^2}{ds} \right)^2 \right) ds^2 - (x^1)^2 dt^2,$$

which would imply M was Lorentzian surface in L^3. Therefore H must be timelike.

Now we may choose an orthonormal frame $\{e_0, e_1, e_2\}$ of L^3 so that $l =$span$\{e_2\}$ and $H =$span$\{e_0, e_2\}$. Then M is given by

$$\begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0(s) \\ 0 \\ x^2(s) \end{bmatrix} = \begin{bmatrix} x^0(s) \sinh t \\ x^0(s) \cosh t \\ x^2(s) \end{bmatrix}.$$

In this case the first fundamental form is given by

$$\left(-\left(\frac{dx^0}{ds}\right)^2+\left(\frac{dx^2}{ds}\right)^2\right)ds^2+(x^0)^2dt^2,$$

which assures that M is spacelike as long as $x^0(s)\neq 0$ and the given curve $(x^0(s),0,x^2(s))$ is spacelike.

(2) Suppose l is timelike. Choose an orthonormal frame $\{e_0,e_1,e_2\}$ of L^3 so that $l=\text{span}\{e_0\}$. Then all the proper rotations about l are represented by the matrix

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos t & -\sin t\\ 0 & \sin t & \cos t\end{bmatrix},\quad t\in R,$$

with respect to the frame.

Note that H must be timelike in this case because we can prove H is nondegenerate as we did in the proof of (1). A nondegenerate plain that contains timelike vectors must be timelike. Therefore we can choose an orthonormal frame $\{e_0,e_1,e_2\}$ of L^3 so that $l=\text{span}\{e_0\}$ and $H=\text{span}\{e_0,e_1\}$. Then M is given by

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos t & -\sin t\\ 0 & \sin t & \cos t\end{bmatrix}\begin{bmatrix}x^0(s)\\ x^1(s)\\ 0\end{bmatrix}=\begin{bmatrix}x^0(s)\\ x^1(s)\cos t\\ x^1(s)\sin t\end{bmatrix}.$$

In this case the first fundamental form is given by

$$\left(-\left(\frac{dx^0}{ds}\right)^2+\left(\frac{dx^1}{ds}\right)^2\right)ds^2+(x^1)^2dt^2,$$

which assures that M is timelike as long as $x^1(s)\neq 0$ and the given curve $(x^0(s),x^1(s),0)$ is spacelike.

(3) Finally suppose l is lightlike. Choose a null frame $\{A,B,C\}$ of L^3 so that $l=\text{span}\{B\}$. Then all the proper rotations about l are represented by the matrix

$$\begin{bmatrix}1 & 0 & 0\\ -\frac{t^2}{2} & 1 & -t\\ t & 0 & 1\end{bmatrix},\quad t\in R.$$

We claim that H is nondegenerate, and hence timelike.

Suppose H was degenerate. Then the profile curve $x(s)$ could be represented by $f(s)U + g(s)B$ for a unit spacelike vector in H such that $U \cdot B = 0$, and the rotatioanary surface of $x(s)$ about B could be represented by

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{t^2}{2} & 1 & -t \\ t & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ bf(s)+g(s) \\ f(s) \end{bmatrix} = \begin{bmatrix} 0 \\ bf(s)+g(s)-tf(s) \\ f(s) \end{bmatrix}.$$

for some $b \in R$. Then the first fundamental form of M would be given by $(\frac{dg}{ds})^2 ds^2$, which would imply M was degenrate. Hence H must be a nondegenerate plane containg a lightlike vector B, which means it is a timelike plain.

Since H is timelike, we may find a null frame $\{A,B,C\}$, so that l =span$\{B\}$, H =span$\{A,B\}$, and $(a(s),b(s),0)$ is a spacelike curve with $a(s) \neq 0$. Then M is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{t^2}{2} & 1 & -t \\ t & 0 & 1 \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \\ 0 \end{bmatrix} = \begin{bmatrix} a(s) \\ -\frac{t^2}{2}a(s)+b(s) \\ ta(s) \end{bmatrix}.$$

This surface has the metric $2\left(\frac{da}{ds}\frac{db}{ds}\right)ds^2 + a^2dt^2$, which is positive definite as long as $a(s) \neq 0$ and

$$\left(\frac{da}{ds}A + \frac{db}{ds}B\right) \cdot \left(\frac{da}{ds}A + \frac{db}{ds}B\right) = \frac{da}{ds}^2 + \frac{db}{ds}^2$$
$$\neq 0.$$

This completes the proof.

Let M be a rotatory spacelike surface in L^3 defined by (1).

If $\dfrac{dx^2}{ds} = 0$ at a point, then M cannot be a spacelike surface. Hence we may assume $\dfrac{dx^2}{ds}$ is nowhere zero on some interval I so that the curve $\alpha(s)$ can be reparametrized by

$$\alpha(s) = (x(s), 0, s),$$

where $x(s)$ is nowhere zero on the interval I. The corresponding surface is given by

$$(x(s)\cosh t, x(s)\sinh t, s)$$

with respect to an orthonormal frame in L^3. Since we want to make M spacelike, we assume $\left(\frac{dx}{ds}\right)^2 < 1$. Then

$$H \equiv 0 \quad \text{if and only if} \quad \frac{d^2x}{ds^2}x - \left(\frac{dx}{ds}\right)^2 + 1 = 0.$$

Let M be a rotatory spacelike surface in L^3 defined by (2). Then M is given by

$$(s, x(s)\cos t, x(s)\sin t),$$

where $x(s)$ is nowhere zero and $\left(\frac{dx}{ds}\right)^2 > 1$. Then

$$H \equiv 0 \quad \text{if and only if} \quad \frac{d^2x}{ds^2}x - \left(\frac{dx}{ds}\right)^2 + 1 = 0.$$

Let M be a rotatory spacelike surface in L^3 defined by (3). Then M is given by

$$\left(a(s), -\frac{t^2}{2} - a(s) + s, ta(s)\right),$$

with respect to a null frame $\{A, B, C\}$, where $a(s)$ is nowhere zero and $\frac{da}{ds} > 0$ everywhere. Then

$$H \equiv 0 \quad \text{if and only if} \quad \frac{d^2a}{ds^2}a + 2\left(\frac{da}{ds}\right)^2 = 0.$$

To obtain all rotatory maxiaml spacelike surfaces in L^3, we need to solve the differential equations.

LEMMA 2. *Let $x(s)$ be a smooth function on I. Then*
(1) The equation

$$\frac{d^2x}{ds^2}x - \left(\frac{dx}{ds}\right)^2 + 1 = 0$$

has the solutions

$$\begin{cases} x(s) = \frac{\sin(cs+d)}{c}, & \text{if} \quad x \neq 0, \quad \left(\frac{dx}{ds}\right)^2 < 1, \\ x(s) = \frac{\sinh(cs+d)}{c}, & \text{if} \quad x \neq 0, \quad \left(\frac{dx}{ds}\right)^2 > 1. \end{cases}$$

(2) The equation

$$\frac{d^2x}{ds^2}x + 2\left(\frac{dx}{ds}\right)^2 = 0$$

has the solutions

$$x(s) = (c^2 s + d)^{1/3} \quad \text{if} \quad x(s) \neq 0, \quad \left(\frac{dx}{ds}\right)^2 > 0.$$

Proof. Let $p = \frac{dx}{ds}$. Then the equations may be reduced to

$$\begin{cases} \left(\dfrac{dp}{dx}\right) px - p^2 + 1 = 0 \\ \left(\dfrac{dp}{dx}\right) px + 2p^2 = 0 \end{cases}$$

or

$$\begin{cases} \dfrac{pdp}{p^2 - 1} = \dfrac{dx}{x} \\ \dfrac{dp}{p} = -2\dfrac{dx}{x}. \end{cases}$$

By integrating both sides we obtain the results easily.

Now the theorem follows immediately.

3. Local formulas

In the section we develop the geometry of space-like hypersurfaces of Lorentzian manifolds using the method of moving frames.

Let N be an $n+1$ dimensional Lorentzian manifold. Let $e_0, \cdots, e_n$ be a local orthonormal frame field in N, and let $\omega_0, \cdots, \omega_n$ be the dual coframe. We shall use the summation convention with Roman indices in the range $1 \leq i, j, \cdots \leq n$ and $0 \leq \alpha, \beta, \cdots \leq n$. Then we have

$$\omega_\alpha(e_\beta) = \delta_{\alpha\beta}$$

and the Lorentzian metric takes the form

$$ds^2 = \sum_\alpha \epsilon_\alpha \omega_\alpha^2,$$

where $\epsilon_\alpha = \pm 1$ according to the signatures of e_α's in N.

PROPOSITION. *There exist 1 forms $\omega_{\alpha\beta}$, and 2 forms $\Omega_{\alpha\beta}$, called connection forms, and curvature forms, determined uniquely by the structure equations of N given by*

$$d\omega_\alpha = -\sum_\beta \epsilon_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \qquad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0, \tag{5}$$

$$d\omega_{\alpha\beta} = -\sum_\gamma \epsilon_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}. \tag{6}$$

Proof. Let D be the Levi-Civita connection defined on N. Define

$$\omega_{\alpha\beta} = \sum_\gamma \epsilon_\gamma \Gamma^\alpha_{\gamma\beta} \omega_\gamma,$$

where

$$D_{e_\alpha} e_\beta = \sum_\gamma \Gamma^\gamma_{\alpha\beta} e_\gamma.$$

These $\omega_{\alpha\beta}$ are the unique 1-forms satisfying the structure equations. The curvature 2-forms $\Omega_{\alpha\beta}$ are then uniquely defined by the equation.

Let K be the Lorentzian curvature tensor on N, and let

$$K(e_\gamma, e_\delta)e_\beta = \sum_\alpha K_{\alpha\beta\gamma\delta} e_\alpha.$$

Then

$$\Omega_{\alpha\beta} = \frac{1}{2} \sum_{\gamma,\delta} \epsilon_\gamma K_{\alpha\beta\gamma\delta} \omega_\gamma \wedge \omega_\delta, \qquad \text{and}$$
$$K_{\alpha\beta\gamma\delta} + K_{\alpha\beta\delta\gamma} = 0.$$

We restrict these forms to M. Then

$$\omega_0 \equiv 0. \tag{7}$$

By using (5)-(7), we obtain

$$(8) \qquad d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$

$$(9) \qquad d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \Theta_{ij},$$

where Θ_{ij} denotes the curvature forms on M.

Let R be the curvature tensor of M given by

$$R(e_k, e_l)e_j = \sum_i R_{ijkl} e_i.$$

Then

$$\Theta_{ij} = \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l.$$

The form $II = \sum_{ij} h_{ij}\omega_i\omega_j$, and the scalar $H = \left(\frac{1}{n}\right) \sum_i h_{ii}$ are called the second fundamental form and the mean curvature of M. Since $0 = d\omega_0 = -\sum_i \omega_{0i} \wedge \omega_i$, by Cartan's lemma, we can write

$$(10) \qquad \omega_{0i} = \sum_j h_{ij}\omega_j, \qquad h_{ij} = h_{ji}.$$

Using (6) and (9),we obtain the Gauss formula

$$R_{ijkl} = -(h_{ik}h_{jl} - h_{il}h_{jk}) + K_{ijkl}.$$

The first covariant derivative of II is defined by

$$(\nabla II)(e_i, e_j) = \sum_k h_{ijk}\omega_k$$
$$= dh_{ij} - \sum_k h_{ik}\omega_{kj} - \sum_k h_{jk}\omega_{ki}.$$

Then, by exterior differentiating (10), we obtain the Coddazi equation

$$(12) \qquad h_{ijk} - h_{ikj} = K_{0ijk}.$$

Next define the second covariant derivative of II by

$$(\nabla^2 II)(e_i, e_j, e_k) = \sum_l h_{ijkl}\omega_l$$
$$= dh_{ijk} - \sum_l h_{ljk}\omega_{lj} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ijl}\omega_{lk}$$

and exterior differentiate (10) to obtain the Ricci formula

$$(13) \qquad h_{ijkl} - h_{ijlk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{jm}R_{mikl}.$$

Let us now define the covariant derivative of K, as a curvature tensor of N,by

$$(DK)(e_i, e_j, e_k, e_l) = \sum_m K_{ijkl,m}\omega_m.$$

Then restricting to M, we obtain

$$(14) \qquad K_{0ijk;l} = K_{oijkl} - h_{jl}K_{0i0k} - h_{kl}K_{0ij0} + \sum_m h_{ml}K_{mijk},$$

where K_{0ijkl} denote the components of the covariant derivative of $\sum_{j,k,l} K_{0jkl}\omega_j\omega_k\omega_l$ so that

$$\sum_l K_{0ijkl}\omega_l = dK_{0ijk} - \sum_m K_{0mjk}\omega_{mi} - \sum_m K_{0imk}\omega_{mj} - \sum_m K_{0ijm}\omega_{mk}.$$

The Laplacian ΔII of the second fundamental form II is defined by

$$\Delta II(e_i, e_j) = \sum_k h_{ijkk}.$$

From (12), we obtain

$$(15) \qquad (\Delta II)(e_i, e_j) = \sum_k \{h_{ikjk} - K_{0ijkk}\} = \sum_k \{h_{kijk} - K_{0ijkk}\}.$$

Also, from (13) we obtain

$$(16) \qquad h_{kijk} = h_{kikj} + \sum_m \{h_{km}R_{mijk} + h_{im}R_{mkjk}\}.$$

Then if we replace h_{kikj} in by $h_{kkij} - K_{0kikj}$ (by 12) and if we substitute the right hand side of (16) into h_{kijk} of (15), we obtain

(17)
$$(\Delta II)(e_i, e_j) = \sum_k \{h_{kkij} - K_{0kikj} - K_{0ijkk}\} + \sum_k \{\sum_m h_{km} R_{mijk} + \sum_m h_{im} R_{mkjk}\}.$$

From (11),(14) and (17) we then obtain

(18)
$$\begin{aligned}\Delta II(e_i, e_j) = &\sum_k \{h_{kkij} + K_{0kik;j} + K_{0ijk,k}\} \\ &+ \sum_k \{h_{kk} K_{0ij0} + h_{ij} K_{0k0k}\} \\ &+ \sum_{m,k} \{h_{mj} K_{mkik} + 2h_{mk} K_{mijk} + h_{mi} K_{mkjk}\} \\ &- \sum_{m,k} \{h_{mi} h_{mj} h_{kk} + h_{km} h_{mj} h_{ik} \\ &\qquad - h_{km} h_{mk} h_{ij} - h_{mi} h_{mk} h_{kj}\}.\end{aligned}$$

4. Maximal spacelike hypersurfaces in Lorentzian manifold with constant curvature

Now we assume that N has constant curvature c and that M is maximal in N, so that $\sum_k h_{kk} = 0$. Then

$$K_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

and

$$R_{ij} = \sum R_{ikjk} = c(n-1)\delta_{ij} + \sum_k h_{ik} h_{kj}. \tag{19}$$

Then easily we know that $((n-1)c\delta_{ij}) \leq (R_{ij})$ and the equality holds everywhere if and only if M is totally geodesic in N.

Now we have the Gauss formula

$$(20) \qquad Rijkl = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk})$$

and Codazzi equation

$$(21) \qquad h_{ijk} - h_{ikj} = 0$$

and the Ricci formula

$$(22) \qquad h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{mi}R_{mjkl}.$$

Note that $K_{\alpha\beta\delta\gamma,\epsilon} \equiv 0$ and $\sum_{i,j,k} h_{kkij}h_{ij} \equiv 0$. Hence

$$(23) \qquad (\Delta II)(e_i, e_j) = \sum_k h_{kkij} + nch_{ij} + Sh_{ij}$$

and

$$(24) \qquad \sum_{i,j} h_{ij}\Delta II(e_i, e_j) = (nc + S)S,$$

where $S = \sum_{i,j} {h_{ij}}^2$ is the length squared of the second fundamental form.

A formula for the Laplacian of S will be needed later. This was first derived by Calabi [3] in the case $N = L^{n+1}$. The works of Cheng and Yau [4] and Treibergs [13] are also relevant here. Nishikawa [10] has used similar computation when N is locally symmetric spacetime with nonnegative spacelike sectional curvature.

$$\begin{aligned}\frac{1}{2}\Delta S &= \sum_{i,j,k} (h_{ijk})^2 + \sum_{i,j} h_{ij}(\Delta II)(e_i, e_j)\\ &= \sum_{i,j,k} (h_{ijk})^2 + (nc + S)S\\ &\geq (nc + S)S.\end{aligned}$$

THEOREM 2. *Let M be a complete maximal spacelike hypersurface in a Lorentzian (n+1)-dimensional manifold N with constant curvature c.*

i) If $c \geq 0$, then M is totally geodesic.

ii) If $c < 0$, and the norm of the second fundamental form is constant, then either M is totally geodesic, or $S = -cn$.

We need the following theorem of [11] to prove the theorem.

THEOREM. (Omori-Yau) *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function which is bounded from below on M. Then for all $\epsilon > 0$ there exists a point x in M such that, at x,*

$$\|\mathrm{grad} f\| < \epsilon, \quad \Delta f > -\epsilon, \quad \text{and} \quad f(x) < \inf f + \epsilon.$$

LEMMA 1. $S \equiv 0$ *or* $S \leq -cn$.

Proof. Note that M satisfies the hypothesis of the Theorem by Omori-Yau. Let's use the maximum principle argument as in [14]. Put $f = 1/\sqrt{S+a}$ for any positive constant a. Then f is a bounded C^∞-function on M. Now we have

$$\Delta f = -\frac{f^3}{2}\Delta S + 3f^5\|\mathrm{grad} S\|^2.$$

Let ϵ be any positive number. Then there is a point x in M such that, at x,

$$\frac{f^6}{4}\|\mathrm{grad} S\| < \epsilon, \quad \Delta f > -\epsilon, \quad \text{and} \quad f(x) < \inf f + \epsilon.$$

Therefore we obtain

$$\frac{f^4}{2}\Delta S < \epsilon(\inf f + \epsilon) + 12\epsilon.$$

Since $\frac{1}{2}\Delta S \geq ncS + S^2$, it follows that

$$\frac{1}{(S+a)^2}\left(-ncS - S^2\right) \geq \frac{1}{(S+a)^2}\cdot\left(-\frac{1}{2}\Delta S\right) \geq -\epsilon(\inf f + \epsilon) - 12\epsilon.$$

When $\epsilon \to 0$, $f(x)$ goes to the infimum and $S(x)$ goes to the supremum. Thus we conclude that the function S is bounded on M, and that if $S \neq 0$ then $S \leq -nc$.

For the proof of the next lemma, see [9].

LEMMA 2. *Suppose $c < 0$. If the norm $| II |$ of the second fundamental form of M is constant, and II does not vanish identically, then $S = -nc$.*

Now we are ready to prove the Theorem. Suppose $c \geq 0$. For any $x \in M$, either $S(x) = 0$ or $S(x) \leq -nc$. Since $S(x) \geq 0$, $S(x) = 0$. Thus i) is proved.

Suppose $c < 0$, and S is constant. Then either $S = 0$ or $S = -nc$. Hence ii) is proved.

References

1. K. Akutagawa, *On spacelike hypersurfaces with constant mean curvature in the de Sitter space*, Math Z. **196** (1986), 13–19.
2. J. K. Beem and P. E. Ehrlich, *Global Lorentzian geometry*, Marcel Dekker, 1981.
3. E. Calabi, *Examples of Bernstein problems for some nonlinear equations*, Proc. Symp. Pure Appl. Math. **15** (1970), 223–230.
4. S. Y. Cheng and S. T. Yau, *Maximal space-like hypersurfaces in the Lorentz–Minkowski spaces*, Ann. Math. **104** (1976), 407–419.
5. S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure and Applied Math. **28** (1975), 333–354.
6. Y. Choquet-Bruhat, A. E. Fischer and J. E. Marsden, *Maximal hypersurfaces and positivity of mass*, Proceedings of the Enrico Fermi Summer School of the Italian Physical Society, J. Ehlers (ed), North-Holland, 1979.
7. S. W. Hawking and G. F. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.
8. T. Ishihara, *Maximal spacelike submanifolds of a pseudoRiemannian space of constant curvature*, Michigan Math. J. **35** (1988), 345–352.
9. U. K. Ki, H. Y. Kim and H. Nakagawa, *Complete maximal space-like hypersurfaces of an Anti-de Sitter space*, (to appear in Kyungpook Math J.).
10. S. Nishikawa, *On maximal spacelike hypersurfaces in a Lorentzian manifold*, Nagoya Math. J. **95** (1984), 117–124.
11. H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc Japan **19** (1967), 205–214.
12. R. Schoen, L. Simon and S. T. Yau, *Curvature estimates for minimal hypersurfaces*, Acta Math. **134** (1975), 275–288
13. A. E. Treibergs, *Entire spacelike hypersurfaces of constant mean curvature in Minkowski space*, Invent. Math. **66** (1982), 39–56.
14. S. T. Yau, *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. **100** (1978), 197–203.

Department of Mathematics
Pusan National University
Pusan 609–735, Korea