

## ON SOME PROPERTIES OF INITIAL CONVERGENCE STRUCTURES

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### 1. Introduction

A convergence structure defined by Kent [4] is a correspondence between the filters on a given set  $X$  and the subsets of  $X$  which specifies which filters converge to which points of  $X$ . This concept is defined to include types of convergence which are more general than that defined by specifying a topology on  $X$ . Thus, a convergence structure may be regarded as a generalization of a topology.

With a given convergence structure  $q$  on a set  $X$ , Kent introduced associated convergence structures which are called a topological modification, a pretopological modification and a pseudotopological modification.

Carstens and Kent [2] seek an answer to the following question: When is the pretopological modification associated with a product of convergence structures equal to the product of the pretopological modifications associated with the factor convergence structures? A pair of convergence spaces  $X, Y$  is said to be topological coherent if the product of their topological modifications is the topological modification of their product. Kent and Richardson [8] devoted to the study of topological coherence. In discussing products of convergence structures, they restricted themselves to finite products. Kent defined the order relation on the set of all convergence structures on  $X$ .

In this paper, we shall study some properties of a convergence structure called the initial convergence structure induced by given convergence structures and functions, in particular, relations between the modifications associated with the initial convergence structure and the initial convergence structures induced by the modifications associated with the factor convergence structures, and initial invariant properties. By the initial convergence structure, the product convergence structure

for a finite family of convergence structures is extended to the product convergence structure for an arbitrary family of convergence structures.

## 2. Preliminaries

A *convergence structure*  $q$  on a set  $X$  is defined to be a function from the set  $F(X)$  of all filters on  $X$  into the set  $P(X)$  of all subsets of  $X$ , satisfying the following conditions:

- (1)  $x \in q(\dot{x})$  for all  $x \in X$ ;
- (2)  $\Phi \subset \Psi$  implies  $q(\Phi) \subset q(\Psi)$ ;
- (3)  $x \in q(\Phi)$  implies  $x \in q(\Phi \cap \dot{x})$ ,

where  $\dot{x}$  denotes the ultrafilter containing  $\{x\}$ ;  $\Phi$  and  $\Psi$  are in  $F(X)$ . Then the pair  $(X, q)$  is called a *convergence space*. If  $x \in q(\Phi)$ , then we say that  $\Phi$  *q-converges* to  $x$ . The filter  $V_q(x)$  obtained by intersecting all filters which  $q$ -converge to  $x$  is called the *q-neighborhood filter* at  $x$ . If  $V_q(x)$   $q$ -converges to  $x$  for each  $x \in X$ , then  $q$  is said to be *pretopological* and the pair  $(X, q)$  is called a *pretopological space*. A convergence structure  $q$  is said to be *pseudotopological* if  $\Phi$   $q$ -converges to  $x$  whenever each ultrafilter finer than  $\Phi$   $q$ -converges to  $x$ , and the pair  $(X, q)$  is called a *pseudotopological space*.

A convergence structure  $q$  is said to be *topological* if  $q$  is pretopological and for each  $x \in X$ , the filter  $V_q(x)$  has a filter base  $B_q(x)$  with the following property:

$$y \in G(x) \in B_q(x) \text{ implies } G(x) \in B_q(y).$$

We shall use the terms “topological convergence structure” and “topology” interchangeably. Also, a pretopological convergence structure and a pseudotopological convergence structure shall usually be called a “pretopology” and a “pseudotopology”, respectively.

Let  $C(X)$  be the set of all convergence structures on  $X$ , partially ordered as follows:

$$q_1 \leq q_2 \text{ iff } q_2(\Phi) \subset q_1(\Phi) \text{ for all } \Phi \in F(X).$$

If  $q_1 \leq q_2$ , then we say that  $q_1$  is coarser than  $q_2$ , and  $q_2$  is finer than  $q_1$ .

For any  $q \in C(X)$ , we define the following related convergence structures,  $\rho(q)$ ,  $\pi(q)$ , and  $\lambda(q)$ :

- (1)  $x \in \rho(q)(\Phi)$  iff  $x \in q(\Phi')$  for each ultrafilter  $\Phi'$  finer than  $\Phi$ .  
 (2)  $x \in \pi(q)(\Phi)$  iff  $V_q(x) \subset \Phi$ .  
 (3)  $x \in \lambda(q)(\Phi)$  iff  $U_q(x) \subset \Phi$ , where  $U_q(x)$  is the filter generated by the sets  $U \in V_q(x)$  which have the property:  $y \in U$  implies  $U \in V_q(y)$ .  
 In this case,  $\rho(q)$ ,  $\pi(q)$  and  $\lambda(q)$  are called the *pseudotopological modification*, the *pretopological modification* and the *topological modification* of  $q$ , and the pairs  $(X, \rho(q))$ ,  $(X, \pi(q))$  and  $(X, \lambda(q))$  are called the *pseudotopological modification*, the *pretopological modification* and the *topological modification* of  $(X, q)$ , respectively.

PROPOSITION 1. ([4])

- (1)  $\rho(q)$  is the finest pseudotopology coarser than  $q$ .  
 (2)  $\pi(q)$  is the finest pretopology coarser than  $q$ .  
 (3)  $\lambda(q)$  is the finest topology coarser than  $q$ .  
 (4)  $\lambda(q) \leq \pi(q) \leq \rho(q) \leq q$ .

Let  $f$  be a map from  $X$  into  $Y$  and  $\Phi$  a filter on  $X$ . Then  $f(\Phi)$  means the filter generated by  $\{f(F) \mid F \in \Phi\}$

Let  $f$  be a map from a convergence space  $(X, q)$  to a convergence space  $(Y, p)$ . Then  $f$  is said to be *continuous* at a point  $x \in X$ , if the filter  $f(\Phi)$  on  $Y$   $p$ -converges to  $f(x)$  for every filter  $\Phi$  on  $X$   $q$ -converging to  $x$ . If  $f$  is continuous at every point  $x \in X$ , then  $f$  is said to be *continuous*. Also,  $f$  is said to be *neighborhood preserving*, if  $V_p(f(x)) = f(V_q(x))$ .

PROPOSITION 2. ([6]) If  $f: (X, q) \rightarrow (Y, p)$  is continuous at  $x \in X$ , then  $V_p(f(x)) \subset f(V_q(x))$ .

Let  $X$  be a nonempty set,  $(X_\lambda, q_\lambda)$  a convergence space and  $f_\lambda: X \rightarrow (X_\lambda, q_\lambda)$  a surjection for each  $\lambda \in \Lambda$ . The *initial convergence structure*  $q$  on  $X$  is defined by specifying that for any  $x \in X$  and  $\Phi \in F(X)$ ,

$$x \in q(\Phi) \text{ iff } f_\lambda(x) \in q_\lambda(f_\lambda(\Phi)) \text{ each } \lambda \in \Lambda.$$

In this case, the pair  $(X, q)$  is called the *initial convergence space* induced by a family of surjections  $\{f_\lambda: X \rightarrow (X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$  and  $q$  is denoted by  $\bigvee_{\lambda \in \Lambda} q_\lambda$ . The initial convergence structure  $q$  is the coarsest convergence structure on  $X$  with respect to which all  $f_\lambda: X \rightarrow (X_\lambda, q_\lambda)$  are continuous ([9]). From now, given a filter  $\Phi_\lambda$  on  $X_\lambda$  for each  $\lambda \in \Lambda$ , suppose that the family  $\{f_\lambda^{-1}(B_\lambda) \mid B_\lambda \in \Phi_\lambda, \lambda \in \Lambda\}$  has the finite

intersection property. Then the *initial filter* of  $\{\Phi_\lambda \mid \lambda \in \Lambda\}$  is the filter on  $X$  which has a base the set of subsets of  $X$  of the form  $\bigcap_{\lambda \in \Lambda'} f_\lambda^{-1}(B_\lambda)$ , where  $B_\lambda \in \Phi_\lambda$  for each  $\lambda \in \Lambda$  and  $\Lambda'$  is a finite subset of  $\Lambda$ . The initial filter of  $\{\Phi_\lambda \mid \lambda \in \Lambda\}$  is denoted by  $\bigvee_{\lambda \in \Lambda} \Phi_\lambda$ . In particular, if  $X = \prod_{\lambda \in \Lambda} X_\lambda$  is the product set and  $f_\lambda: X \rightarrow (X_\lambda, q_\lambda)$  is the  $\lambda$ -th projection, then  $\bigvee_{\lambda \in \Lambda} q_\lambda$  and  $\bigvee_{\lambda \in \Lambda} \Phi_\lambda$  are called the *product convergence structure* and the *product filter*, and are denoted by  $\bigvee_{\lambda \in \Lambda} q_\lambda$  and  $\bigvee_{\lambda \in \Lambda} \Phi_\lambda$ , respectively.

The followings are immediate results of above definitions.

**PROPOSITION 3.** Let  $(X_\lambda, q_\lambda)$  be a convergence space,  $\Phi_\lambda$  a filter on  $X_\lambda$ ,  $f_\lambda: X \rightarrow X_\lambda$  a surjection for each  $\lambda \in \Lambda$ ,  $\Phi$  a filter on  $X$ ,  $x \in X$  and  $(X, q)$  the initial convergence space induced by  $\{f_\lambda: X \rightarrow (X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$ . Then the followings hold:

- (1)  $\Phi_\lambda \subset f_\lambda(\bigvee_{\lambda \in \Lambda} \Phi_\lambda)$ . (2)  $\bigvee_{\lambda \in \Lambda} f_\lambda(\Phi) \subset \Phi$ .
- (3)  $f_\lambda(\bigvee_{\lambda \in \Lambda} f_\lambda(\Phi)) = f_\lambda(\Phi)$ .
- (4)  $\bigvee_{\lambda \in \Lambda} V_{q_\lambda}(f_\lambda(x)) \subset V_q(x)$ . (If  $q_\lambda$  is pretopological for each  $\lambda \in \Lambda$ , then the equality holds)
- (5) If  $\Phi_\lambda$   $q_\lambda$ -converges to  $f_\lambda(x)$  for each  $\lambda \in \Lambda$ , then  $\bigvee_{\lambda \in \Lambda} \Phi_\lambda$   $q$ -converges to  $x$ .

**PROPOSITION 4.** Let  $f_\lambda: X \rightarrow X_\lambda$  be a surjection,  $q_\lambda$  and  $p_\lambda$  convergence spaces on  $X_\lambda$  for each  $\lambda \in \Lambda$ ,  $(X, q)$  and  $(X, p)$  and the initial convergence spaces induced by  $\{f_\lambda: X \rightarrow (X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$  and  $\{f_\lambda: X \rightarrow (X_\lambda, p_\lambda) \mid \lambda \in \Lambda\}$ , respectively. If  $q_\lambda \leq p_\lambda$  for each  $\lambda \in \Lambda$ , then  $q \leq p$ , that is,  $\bigvee_{\lambda \in \Lambda} q_\lambda \leq \bigvee_{\lambda \in \Lambda} p_\lambda$ .

*Proof.* Let  $\Phi \in F(X)$  and  $x \in p(\Phi)$ . Then  $f_\lambda(x) \in p_\lambda(f_\lambda(\Phi)) \subset q_\lambda(f_\lambda(\Phi))$  for any  $\lambda \in \Lambda$ . Thus  $x \in q(\Phi)$ .

### 3. Main results

**PROPOSITION 5.** Let  $(X_\lambda, q_\lambda)$  be a convergence space,  $f_\lambda: X \rightarrow X_\lambda$  a surjection for each  $\lambda \in \Lambda$ ,  $(X, q)$  the initial convergence space induced by  $\{f_\lambda: X \rightarrow (X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$ . Then the followings hold:

- (1) If  $q_\lambda$  is pretopological for each  $\lambda \in \Lambda$ , then  $q$  is pretopological.
  - (2) If  $q_\lambda$  is topological for each  $\lambda \in \Lambda$ , then  $q$  is topological.
- In the case  $f_\lambda$  is neighborhood preserving for each  $\lambda \in \Lambda$ , the converses of (1) and (2) are true.

*Proof.* (1) Suppose that  $q_\lambda$  is pretopological for each  $\lambda \in \Lambda$ . Let  $x$  be any element of  $X$ . Since  $V_{q_\lambda}(f_\lambda(x)) \subset f_\lambda(V_q(x))$  and  $f_\lambda(x) \in q_\lambda(V_{q_\lambda}(f_\lambda(x)))$ ,  $f_\lambda(x) \in q_\lambda(f_\lambda(V_q(x)))$  for each  $\lambda \in \Lambda$ . Thus,  $x \in q(V_q(x))$  and  $q$  is pretopological.

Suppose that  $q$  is pretopological and  $f_\lambda$  is neighborhood preserving for each  $\lambda \in \Lambda$ . Let  $y$  be any element of  $X_\lambda$  and  $y = f_\lambda(x)$  for some  $x \in X$ . Then  $x \in q(V_q(x))$ . Thus,

$$y = f_\lambda(x) \in q_\lambda(f_\lambda(V_q(x))) = q_\lambda(V_{q_\lambda}(f_\lambda(x))) = q_\lambda(V_{q_\lambda}(y))$$

and  $q_\lambda$  is pretopological.

(2) Suppose that  $q_\lambda$  is topological for each  $\lambda \in \Lambda$ . Let  $x \in X$  and  $f_\lambda(x) = x_\lambda$ . Then the filter  $V_{q_\lambda}(x_\lambda)$  has a filter base  $B_{q_\lambda}(x_\lambda)$  with the following property:  $y_\lambda \in G_\lambda \in B_{q_\lambda}(x_\lambda)$  implies  $G_\lambda \in B_{q_\lambda}(y_\lambda)$ . Let  $B_q(x)$  be the family of subsets of  $X$  of the form  $\bigcap_{\lambda \in \Lambda'} f_\lambda^{-1}(B_\lambda)$ , where  $B_\lambda \in B_{q_\lambda}(x_\lambda)$  and  $\Lambda'$  is a finite subset of  $\Lambda$ . Then  $B_q(x)$  is the filter base for the initial filter  $\bigvee_{\lambda \in \Lambda} V_{q_\lambda}(x_\lambda)$ . Since  $q_\lambda$  is pretopological,  $\bigvee_{\lambda \in \Lambda} V_{q_\lambda}(x_\lambda) = V_q(x)$ . Therefore,  $B_q(x)$  is the filter base for  $V_q(x)$ . Let  $y \in G \in B_q(x)$  and  $G = \bigcap_{\lambda \in \Lambda'} f_\lambda^{-1}(B_\lambda)$ , where  $B_\lambda \in B_{q_\lambda}(x_\lambda)$  and  $\Lambda'$  is a finite subset of  $\Lambda$ . Then  $y \in f_\lambda^{-1}(B_\lambda)$  and hence  $f_\lambda(y) \in B_\lambda$  for each  $\lambda \in \Lambda'$ . Thus,  $B_\lambda \in B_{q_\lambda}(f_\lambda(y))$  and  $G \in B_q(y)$ . Consequently,  $q$  is topological.

Conversely, suppose that  $q$  is topological and  $f_\lambda$  is neighborhood preserving for each  $\lambda \in \Lambda$ . Then for each  $x \in X$ ,  $V_q(x)$  has a filter base  $B_q(x)$  with the following property:  $y \in G \in B_q(x)$  implies  $G \in B_q(y)$ . Let  $B_{q_\lambda}(f_\lambda(x)) = \{f_\lambda(G) \mid G \in B_q(x)\}$ . Then  $B_{q_\lambda}(f_\lambda(x))$  is a filter base for  $f_\lambda(V_q(x)) = V_{q_\lambda}(f_\lambda(x))$  since  $f_\lambda$  is neighborhood preserving. Let  $y_\lambda \in G_\lambda \in B_{q_\lambda}(f_\lambda(x))$ . Then there exists  $G \in B_q(x)$  such that  $f_\lambda(G) = G_\lambda$ . Take  $y \in G$  with  $f_\lambda(y) = y_\lambda$ . Then  $G \in B_q(y)$  and hence  $G_\lambda = f_\lambda(G) \in B_{q_\lambda}(f_\lambda(y)) = B_{q_\lambda}(y_\lambda)$ . Therefore,  $q_\lambda$  is topological.

**PROPOSITION 6.** Let  $f_\lambda: X \rightarrow X_\lambda$  be a surjection and  $(X_\lambda, \rho(q_\lambda))$  the pseudotopological modification of a convergence space  $(X_\lambda, q_\lambda)$  for each  $\lambda \in \Lambda$ . Let  $(X, \bigvee_{\lambda \in \Lambda} q_\lambda)$  and  $(X, \bigvee_{\lambda \in \Lambda} \rho(q_\lambda))$  be the initial convergence spaces induced by  $\{f_\lambda: X \rightarrow (X, q_\lambda) \mid \lambda \in \Lambda\}$  and  $\{f_\lambda: X \rightarrow (X_\lambda, \rho(q_\lambda)) \mid \lambda \in \Lambda\}$ , respectively. Then  $\rho(\bigvee_{\lambda \in \Lambda} q_\lambda) \leq \bigvee_{\lambda \in \Lambda} \rho(q_\lambda)$ .

*Proof.* Let  $\Phi$  be a filter on  $X$  and  $x \in \bigvee_{\lambda \in \Lambda} \rho(q_\lambda)(\Phi)$ . Then  $f_\lambda(x) \in \rho(q_\lambda)(f_\lambda(\Phi))$  for each  $\lambda \in \Lambda$ . Let  $\Phi'$  be an ultrafilter finer than  $\Phi$ . Then

$f_\lambda(\Phi') \supset f_\lambda(\Phi)$  and  $f_\lambda(\Phi')$  is an ultrafilter on  $X_\lambda$ . Thus,  $f_\lambda(x) \in q_\lambda(f_\lambda(\Phi'))$  for each  $\lambda \in \Lambda$ . Therefore,  $x \in (\bigvee_{\lambda \in \Lambda} q_\lambda)(\Phi')$  and  $x \in \rho(\bigvee_{\lambda \in \Lambda} q_\lambda)(\Phi)$ . Consequently,  $(\bigvee_{\lambda \in \Lambda} \rho(q_\lambda))(\Phi) \subset \rho(\bigvee_{\lambda \in \Lambda} q_\lambda)(\Phi)$ , that is,  $\rho(\bigvee_{\lambda \in \Lambda} q_\lambda) \leq \bigvee_{\lambda \in \Lambda} \rho(q_\lambda)$ . This completes the proof.

Consequently, we obtain the following corollary.

**COROLLARY 7.** *Let  $f_\lambda: X \rightarrow X_\lambda$  be a surjection and  $(X_\lambda, \lambda(q_\lambda))$ ,  $(X_\lambda, \pi(q_\lambda))$  and  $(X_\lambda, \rho(q_\lambda))$  be the topological, the pretopological and the pseudotopological modification of a convergence space  $(X_\lambda, q_\lambda)$  for each  $\lambda \in \Lambda$ , respectively. Let  $(X, \bigvee_{\lambda \in \Lambda} q_\lambda)$ ,  $(X, \bigvee_{\lambda \in \Lambda} \lambda(q_\lambda))$ ,  $(X, \bigvee_{\lambda \in \Lambda} \pi(q_\lambda))$  and  $(X, \bigvee_{\lambda \in \Lambda} \rho(q_\lambda))$  be the initial convergence spaces induced by*

$$\{f_\lambda: X \rightarrow (X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}, \{f_\lambda: X \rightarrow (X_\lambda, \lambda(q_\lambda)) \mid \lambda \in \Lambda\},$$

$$\{f_\lambda: X \rightarrow (X_\lambda, \pi(q_\lambda)) \mid \lambda \in \Lambda\} \text{ and } \{f_\lambda: X \rightarrow (X_\lambda, \rho(q_\lambda)) \mid \lambda \in \Lambda\},$$

respectively. Then the following inequalities hold:

$$(1) \bigvee_{\lambda \in \Lambda} \lambda(q_\lambda) \leq \lambda(\bigvee_{\lambda \in \Lambda} q_\lambda) \leq \pi(\bigvee_{\lambda \in \Lambda} q_\lambda).$$

$$(2) \bigvee_{\lambda \in \Lambda} \lambda(q_\lambda) \leq \bigvee_{\lambda \in \Lambda} \pi(q_\lambda) \leq \pi(\bigvee_{\lambda \in \Lambda} q_\lambda) \\ \leq \bigvee_{\lambda \in \Lambda} \rho(q_\lambda) \leq \bigvee_{\lambda \in \Lambda} q_\lambda.$$

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