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ON SLICES

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0. Introction

We will use the following notational conventions : a "product space" will always mean a product topological space equipped with the Tychonoff product topology ; given a product space $\left(\prod_{j\in J} X_j,\ \langle T_j\rangle_{j\in J}\right)$, p_j will denote the projection of the product onto the factor X_j.

In [7], Dugundji introduced the notion of slice as a line parallel to a factor space through a point of a cartesian product space (See §2 for precise definition).

The purpose of this note is to study some properties of slices. This note is neither intended to present substantial new results nor provide an encyclopaedic survey, but rather to give certain aspect to the subject for the choice of personal taste and preference.

1. Preliminaries and notations

Let a mapping $f : S \to X$ be given. By the canonical extension of f, we mean an induced mapping $\overrightarrow{f} : \mathcal{P}(S) \to \mathcal{P}(X)$ by f with the property :

$$\overrightarrow{f}(A) = \{f(a) | a \in A\} \quad \text{for each} \quad A \in \mathcal{P}(S),$$

where $\mathcal{P}(S)$ is the power set of S.

By the f-inverse image mapping, we mean an induced mapping $\overleftarrow{f} : \mathcal{P}(X) \to \mathcal{P}(S)$ by f with the property :

$$\overleftarrow{f}(B) = \{x \in X | f(x) \in B\} \quad \text{for each} \quad B \in \mathcal{P}(X).$$

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For two mappings $f : S \to X$ and $g : S' \to X'$, by the equality of f and g, denoted by $f = g$, we mean that $S = S'$, $X = X'$, and $f(a) = g(a)$ for each $a \in S$.

Note that for each mapping $f : S \to X$, each of the following statements is true :

(1) For each $B \in \mathcal{P}(X \setminus \overrightarrow{f}(S))$, $\overleftarrow{f}(B) = \emptyset$.

(2) For each $B \in \mathcal{P}(X)$, $\overrightarrow{f} \circ \overleftarrow{f} = B \cap \overrightarrow{f}(S)$ and $\overrightarrow{f} \circ \overleftarrow{f}(B) \subset B$.

(3) For each $A \in \mathcal{P}(S)$, $A \subset \overleftarrow{f} \circ \overrightarrow{f}(A)$.

(4) For each mapping $g : X \to Y$, $\overrightarrow{g \circ f} = \overrightarrow{g} \circ \overrightarrow{f}$ and $\overleftarrow{g \circ f} = \overleftarrow{f} \circ \overleftarrow{g}$.

(5) f is surjective if and only if $\overrightarrow{f}(S) = X$ if and only if $\overrightarrow{f} \circ \overleftarrow{f} = 1_{\mathcal{P}(X)}$ where $1_{\mathcal{P}(X)}$ is the identity mapping on $\mathcal{P}(X)$.

By the inclusion mapping of $A \subset S$ into S, denoted by i_A, we mean the mapping $i_A : A \to S$ defined by $i_A(a) = a$ for each $a \in A$, so that for each $E \subset S$, $\overrightarrow{i_A}(E) = E \cap A$ and $\overleftarrow{i_A}(E) = E \cap A$.

Any mapping with having the empty domain is called empty.

Note that for each mapping $f : S \to X$ and each $A \subset S$, each of the following statements is true :

(1) For the restriction $f\big|_A$ of f to A, $\overrightarrow{f\big|_A} = \overrightarrow{f} \circ \overrightarrow{i_A}$ and $\overleftarrow{f\big|_A} = \overleftarrow{i_A} \circ \overleftarrow{f}$.

(2) For each $E \subset S$ and each $F \subset X$, $\overrightarrow{f\big|_A}(E) = \overrightarrow{f\big|_A}(A \cap E)$ and $\overleftarrow{f\big|_A}(F) = A \cap \overleftarrow{f}(F)$.

(3) For each $B \subset X$ and each $F \subset X$,

$$\overleftarrow{f\big|_{\overleftarrow{f}(B)}}(F) = \overleftarrow{f\big|_{\overleftarrow{f}(F)}}(B) = \overleftarrow{f\big|_{\overleftarrow{f}(F)}}(B \cap F) = \overleftarrow{f\big|_{\overleftarrow{f}(B)}}(B \cap F).$$

Let $(A_j)_{j \in J}$ be a family of non-empty sets, let p_j be a projection of the cartesian product $\prod_{j \in J} A_j$ onto A_j, and let X_j be a non-empty subset of A_j. A subset $\overleftarrow{p_j}(X_j)$ of $\prod_{j \in J} A_j$ is called a slab of X_j in $\prod_{j \in J} A_j$.

Note that each of the following statements is easily verified :

(1) $f \in \overleftarrow{p_j}(X_j)$ if and only if $f \in \prod_{j \in J} A_j$ and $f(j) \in X_j$.

(2) $\prod_{j \in J} X_j = \bigcap_{j \in J} \overleftarrow{p_j}(X_j)$.

(3) $\prod_{j \in J} A_j \setminus \overleftarrow{p_j}(X_j) = \overleftarrow{p_j}(A_j \setminus X_j)$.

(4) $\prod_{j\in J} A_j \setminus \prod_{j\in J} X_j = \bigcup_{j\in J} \overleftarrow{p_j}(A_j \setminus X_j)$.

For the sake of later use, we give the following.

LEMMA 1. *For each family $(A_j)_{j\in J}$ of non-empty sets and each non-empty subset K of the index set J, any mapping $P_K : \prod_{j\in J} A_j \to \prod_{k\in K} A_k$ defined by $P_K(f) = f\big|_K$ for each $f \in \prod_{j\in J} A_j$ is surjective.*

Proof. If $K = J$, then P_K is the identity mapping, so our Lemma is true. Let $K \neq J$ and let $g \in \prod_{k\in K} A_k$; we are going to find an $f \in \prod_{j\in J} A_j$ such that $P_K(f) = g$. Since $(A_j)_{j\in J\setminus K}$ is a family of non-empty sets, we can find a choice function s for $(A_j)_{j\in J\setminus K}$, and hence there exists a unique extension f on J of g and s such that $f\big|_K = g$, $f\big|_{J\setminus K} = s$ and $P_K(f) = g$.

Let $(A_j)_{j\in J}$ be a family of non-empty sets, let p_j be the projection of the cartesian product $\prod_{j\in J} A_j$ onto the factor set A_j, and let K be a non-empty subset of the index set J. For each $j \in J$ and each $G \subset A_j$, we define a relative slab of G to $\prod_{k\in K} A_k$, which will be written $\overleftarrow{p_j}(G)\big|_{\prod_{k\in K} A_k}$, by the set such that

$$\overleftarrow{p_j}\big|_{\prod_{k\in K} A_k} = \begin{cases} \prod_{k\in K} A_k & \text{if } j \in J \setminus K, \\ \{f \in \prod_{k\in K} A_k | p_k(f) \in G\} & \text{if } j \in K. \end{cases}$$

2. Definition of slices and basic properties

Let $(A_j)_{j\in J}$ be a family of non-empty sets, let p_j be the projection of the cartesian product $\prod_{j\in J} A_j$ onto the factor set A_j, and let K be a non-empty subset of the index set J.

For each $x \in \prod_{j\in J} A_j$, a subset

$$\bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(x)) = \{f \in \prod_{j\in J} A_j | p_j(f) = p_j(x) \quad \text{for each} \quad j \in J \setminus K\}$$

of $\prod_{j\in J} A_j$ is called a K-slice through the point x parallel to $\prod_{k\in K} A_k$.

Note that each slice is also a cartesian product of sets.

In a set-theoretical sense, we have following.

LEMMA 2. *Let $(A_j)_{j\in J}$ be a family of non-empty sets, let p_j be the projection of $\prod_{j\in J} A_j$ onto A_j, and let K be a subset of the index set J. A K-slice through a point $x \in \prod_{j\in J} A_j$ parallel to $\prod_{k\in K} A_k$ is equippollent to $\prod_{k\in K} A_k$.*

Proof. Let $\bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(x))$ be the K- slice through the point $x \in \prod_{j\in J} A_j$ parallel to $\prod_{k\in K} A_k$. Define a mapping

$$F : \bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(x)) \longrightarrow \prod_{k\in K} A_k$$

to satisfy $F(g) = g\big|_K$ for each $g \in \bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(x))$. We are going to show that F is bijective. Noting that each $g \in \bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(x))$ has the property such that $g(j) = p_j(x)$ for each $j \in J \setminus K$, $F(f) = F(g)$ implies that $f\big|_K = g\big|_K$, from which it follows that $f = g$, and hence F is injective. It remains to prove that F is surjective. To this end, Let $a \in \prod_{k\in K} A_k$, then since $s = (p_j(x))_{j\in J\setminus K}$ is considered as a choice function for $\{\{p_j(x)\} \mid j \in J \setminus K\}$, we can find an extension f on J of a and s such that $f\big|_K = a$ and $f\big|_{J\setminus K} = s$, showing taht $F(f) = a$, from which it follows that F is surjective.

A mapping F mentioned above in the proof will be called a slice bijection.

The K-slice through $x \in \prod_{j\in J} A_j$ equipped with the relativized topology with respect to the Tychonoff product topology for $\prod_{j\in J} A_j$ is called the K-slice space through the point $x \in \prod_{j\in J} A_j$.

THEOREM 1. *Let $(A_j, \mathcal{T}_j)_{j\in J}$ be a family of non-empty topological spaces, let $\langle \mathcal{T}_j \rangle_{j\in J}$ be the Tychonoff topology on the cartesian product $\prod_{j\in J} A_j$, and let p_j be the projection of the space $\prod_{j\in J} A_j$ onto the factor space $(A_j, \mathcal{T}_j)$. Then each K-slice space $\bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(x))$ through an $x \in \prod_{j\in J} A_j$ is homeomorphic to the space $(\prod_{k\in K} A_k, \langle \mathcal{T}_k \rangle_{k\in K})$.*

Proof. Noting that each subbasic open set of the slice space $\bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(x))$ is denoted by a set $\overleftarrow{p_m}(G) \cap (\bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(x)))$ for some $m \in J$ and open G in the space $(A_m, \mathcal{T}_m)$, and that each subbasic open set of the product space $(\prod_{k\in K} A_k, \langle \mathcal{T}_k\rangle_{k\in K})$ is denoted by a set of form $\overleftarrow{p_m}(G)|_{\prod_{k\in K} A_k}$ for some $m \in K$ and open G in the space $(A_m, \mathcal{T}_m)$, the slice bijection $F : \bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(x)) \to \prod_{k\in K} A_k$ gives that $\overrightarrow{F}((\overleftarrow{p_m}(G) \cap (\bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(x))) = \overleftarrow{p_m}(G)|_{\prod_{k\in K} A_k}$, from which it follows that F has further properities of continuity and openness, establishing that F is a homeomorphism.

Letting K be a singleton $\{k\}$, we have the following

COROLLARY. *Each $\{k\}$-slice space $\bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(x))$ is homeomorphic to the space $(A_k, \mathcal{T}_k)$.*

By a diagonal extension of a family $(f_j : X \to A_j)_{j\in J}$ of mappings into a cartesian proproduct $\prod_{j\in J} A_j$, we mean a mapping $\Delta_{j\in J} f_j$: $X \to \prod_{j\in J} A_j$ such that for each projection p_j of $\prod_{j\in J} A_j$ onto A_j, $p_j \circ \Delta_{j\in J} f_j = f_j$.

LEMMA 3. *Let $(A_j, \mathcal{T}_j)_{j\in J}$ be a family of topological spaces, let $\langle \mathcal{T}_j\rangle_{j\in J}$ be the Tychonoff product topology on the cartesian product $\prod_{j\in J} A_j$, and let p_j be the projection of the space $(\prod_{j\in J} A_j, \langle \mathcal{T}_j\rangle_{j\in J})$ onto the factor space $(A_j, \mathcal{T}_j)$. Then the diagonal extension of a family $(f_j : (X, \mathcal{T}) \to (A_j, \mathcal{T}_j))_{j\in J}$ of mappings is continuous if and only if each f_j is continuous.*

Proof. let $\Delta_{j\in J} f_j$ be the diagonal extension of the family $(f_j)_{j\in J}$ of mappings. "Only if" part : Since each p_j is continuous, and since $f_j = p_j \circ \Delta_{j\in J} f_j$ for each $j \in J$, each f_j is continuous if the diagonal extension $\Delta_{j\in J} f_j$ is continuous. "If" part : Let G be $\mathcal{T}_j$-open ; then $\overleftarrow{p_j}(G)$is subbasic open set in the product space $(\prod_{j\in J} A_j, \langle \mathcal{T}_j\rangle_{j\in J})$, and hence

$$(\overleftarrow{\Delta_{j\in J} f_j}) \circ \overleftarrow{p_j}(G) = \overleftarrow{p_j \circ \Delta_{j\in J} f_j}(G) = \overleftarrow{f_j}(G)$$

shows that $\overleftarrow{f_j}(G)$ is an open set in the space $(X, \mathcal{T})$, from which it follows that each f_j is continuous.

LEMMA 4. *Let $(\prod_{j\in J} A_j, \langle \mathcal{T}_j\rangle_{j\in J})$ be a product space of a family $(A_j, \mathcal{T}_j)_{j\in J}$ of spaces, let $\{J_m | m \in M\}$ be a partition of the index* *set J, and for each $m \in M$ let $Y_m = \prod_{j\in J_m} A_j$. Then the product space $(\prod_{j\in J} A_j, \langle \mathcal{T}_j\rangle_{j\in J})$ is homeomorphic to the product space $(\prod_{m\in M} Y_m, \langle \mathcal{Y}_m\rangle_{m\in M})$ where $\mathcal{Y}_m = \langle \mathcal{T}_j\rangle_{j\in J_m}$.*

Proof. Let $m \in M$ and let $S_m : (\prod_{j\in J} A_j, \langle \mathcal{T}_j\rangle_{j\in J}) \to (Y_m, \mathcal{Y}_m)$ be defined by $S_m(f) = f\big|_{J_m}$ for each $f \in \prod_{j\in J} A_j$: then by Lemma 1, S_m is surjective for each $m \in M$. Let G be a subbasic open set in the space $(Y_m, \mathcal{Y}_m) = (\prod_{j\in J_m} A_j, \langle \mathcal{T}_j\rangle_{j\in J_m})$; then we can find a $j \in J_m$ such that $G = \overleftarrow{p_j}(H)\big|_{Y_m}$ with $H \in \mathcal{T}_j$, and hence $\overleftarrow{S_m}(G) = \overleftarrow{p_j}(H)$ is a subbasic open set in $(\prod_{j\in J} A_j, \langle \mathcal{T}_j\rangle_{j\in J})$, so that each S_m is continuous, from which it follows that the diagonal extension $\Delta_{m\in M} S_m$ is continuous by Lemma 3. We are going to show that the diagonal extension $\Delta_{m\in M} S_m$ is an open bijection. To this end, firstly, let $\Delta_{m\in M} S_m(f) = \Delta_{m\in M} S_m(g)$ for each $m \in M$, then $f\big|_{J_m} = g\big|_{J_m}$ for each $m \in M$, from which it follows that the diagonal extension $\Delta_{m\in M} S_m$ is injective.

Secondly, if $g \in \prod_{m\in M} Y_m$ then $g\big|_{J_m} \in (Y_m, \mathcal{Y}_m)$ for each $m \in M$, and hence, since $S_m(g) = g\big|_{J_m}$, we have $(\Delta_{m\in M} S_m)(g) = g$, showing that $\Delta_{m\in M} S_m$ is surjective. Now, it remains to show that $\Delta_{m\in M} S_m$ is open. To this end, let G be a basic open set in $(\prod_{j\in J} A_j, \langle \mathcal{T}_j\rangle_{j\in J})$; then we can find a finite set $K \subset J$ such that $G = \bigcap_{k\in K} \overleftarrow{p_k}(U_k)$ where $U_k \in \mathcal{T}_k$ for each $k \in K$. Noting that $M_k = \{m | K \cap J_m \neq \emptyset\}$ is finite, and that $K \cap J_m \neq \emptyset$ for each $m \in M_k$, $\bigcap_{k\in K\cap J_m} \overleftarrow{p_k}(U_k)$ is a subbasic open set in $(Y_m, \mathcal{Y}_m)$ for each $m \in M_k$, from which it follows that $(\Delta S_m)(G) = \bigcap_{m\in M_k}(\bigcap_{k\in K\cap J_m} \overleftarrow{p_k}(U_k))$ is open set in $(\prod_{m\in M} Y_m, \langle \mathcal{Y}_m\rangle_{m\in M})$.

By Theorem 1 and Lemma 4, we have the following

THEOREM 2. *Let $(A_j, \mathcal{T}_j)_{j\in J}$ be a family of topological spaces, and let $K \subset J$ have a partition $\{K_m | m \in M_j\}$. Then a K-slice space $\bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(x))$ through an $x \in \prod_{j\in J} A_j$ is homeomorphic to $(\prod_{m\in M} Y_m, \langle \mathcal{Y}_m\rangle_{m\in M})$ where $Y_m = \prod_{j\in K_m} A_j$ and $\mathcal{Y}_m = \langle \mathcal{T}_j\rangle_{j\in K_m}$.*

A topological space $(X, \mathcal{T})$ is said to be KV or a KV space whenever it suffices the following [K] and [V] properities of separation axioms of Kolomogoroff and Vietoris, respectively :

[K] For each pair of distinct points, at least one has a $\mathcal{T}$-open neighbourhood not containing the other.
[V] For each point x and each $\mathcal{T}$-open neighbourhood G of x, there exists $\mathcal{T}$-open neighbourhood of x whose $\mathcal{T}$-closure is contained in G.

For invariance properity, we have the following

THEOREM 3.
(1) *Each subspace of a* KV *space is* KV.
(2) *Let* $(\prod_{j\in J} X_j, \langle \mathcal{T}_j \rangle_{j\in J})$ *be a product space of a family* $(X_j, \mathcal{T}_j)_{j\in J}$ *of topological spaces. Then the product space is* KV *if and only if each slice space is* KV.

Proof. (1) Noting that each subspace of a KV space satisfies the conditions [K] and [V], the result follows at once.

(2) "Only if" part : It follows immediately from (1) that if the product space is KV, then each slice space as a subspace of the product space is KV. "If" part : Let each slice space be KV ; then since for each $f \in \prod_{j\in J} X_j$ and each slice space $\bigcap_{j\in J\setminus K} \overleftarrow{p_j}(p_j(f))$ is homeomorphic to a factor space $(X_k, \mathcal{T}_k)$, each factor space $(X_k, \mathcal{T}_k)$ is KV. Let G be a basic open set containing $f \in \prod_{j\in J} X_j$, so that $f \in G = \bigcap_{j\in K} \overleftarrow{p_j}(G_j)$ for some finite $K \subset J$ and $G_j \in \mathcal{T}_j$ for each $j \in K$; then $p_j(f) \in \overrightarrow{p_j}(G) = G_j$ for each $j \in K$. Since each $(X_k, \mathcal{T}_k)$ is KV, we can find a $\mathcal{T}_j$ open neighbourhood of $p_j(f)$ such that $\mathrm{cl}_{\mathcal{T}_j} V_j \subset G_j$, showing that

$$f \in \bigcap_{j\in K} \overleftarrow{p_j}(V_j) \subset \bigcap_{j\in K} \overleftarrow{p_j}(\mathrm{cl}_{\mathcal{T}_j} V_j) \subset \bigcap_{j\in K} p_j(G_j) = G,$$

from which it follows that the product space satisfies the condition [V]. Let $f \neq g$ in the product space ; then for some $k \in J$, $p_k(f) \neq p_k(g)$. Since $(X_k, \mathcal{T}_k)$ is KV, we may assume that there exists a $\mathcal{T}_k$-open neighbourhood W_k of $p_k(g)$ such that $p_k(f) \notin W_k$,from which it follows that $\overleftarrow{p_k}(W_k)$ is a subbasic open set containing g such that $f \notin \overleftarrow{p_k}(W_k)$, showing that the product space satisfies the condition [K].

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