Pusan Kyŏngnam Math. J. 8(1992), No. 2, pp. 251–265

COMMON FIXED POINTS FOR COMPATIBLE MAPPINGS OF TYPE (A)

Y. J. CHO, J. P. KIM AND T. KUBIAK

1. Introduction

The concept of *2-metric spaces* has been investigated initially by S. Gähler in a series of papers ([3]–[5]) and has been developed extensively by S. Gähler and many others.

A 2-metric space is a set X with a real-valued function d on $X \times X \times X$ satisfying the following conditions :

(M_1) For two distinct points x, y in X, there exists a point z in X such that $d(x, y, z) \neq 0$,
(M_2) $d(x, y, z) = 0$ if at least two of x, y, z are equal,
(M_3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$,
(M_4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all x, y, z, u in X.

The function d is called a 2-metric for the space X and (X, d) denotes a 2-metric space. It has been shown by S. Gähler ([3]) that a 2-metric d is non-negative and although d is a continuous function of any one of its three arguments, it need not be continuous in two arguments. A 2-metric d which is continuous in all of its arguments will be called continuous.

On the other hand, a number of authors ([1], [2], [6]–[8], [11]–[12]) have proved many kinds of fixed point theorems in 2-metric spaces. Especially, S.V.R. Naidu and J.R. Prasad ([18]) introduced the concept of weakly commuting pairs of self-mappings on a 2-metric space and the notion of weak continuity of a 2-metric, respectively, and they have proved serveral common fixed point theorems by using the concepts of the weakly commuting pairs of self-mappings on a 2-metric space and the weak continuity of a 2-metric.

Received December 3, 1992

Recently, G. Jungck el al. ([9],[10]) introducted the concepts of compatible mappings and compatible mappings of type (A) in metric spaces and proved some common fixed point theorems for these mappings.

In this paper, we prove some common fixed point theorems for compatible mappings of type (A) in 2-metric spaces. Our results extend, generalize and improve a number of fixed point theorems for commuting and weakly commuting mappings in 2-metric spaces.

2. Compatible mappings of type (A)

In this section, we introduce some definitions, the concepts and some properties of compatible mappings and compatible mappings of type (A) in 2-metric spaces, and show that these mappings are equivalent under some conditions. Throughout this paper, (X, d) denotes a 2-metric space with the continuous 2-metric d.

DEFINITION 2.1. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be *convergent* to a point x in X, which is denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n, x, z) = 0$ for all z in X. The point x is called the *limit* of the sequence $\{x_n\}$ in X.

DEFINITION 2.2. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a *Cauchy sequence* if $\lim_{m,n\to\infty} d(x_m, x_n, z) = 0$ for all z in X.

DEFINITION 2.3. A 2-metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

DEFINITION 2.4. A mapping S from a 2-metric space (X, d) into itself is said to be *sequentially continuous* at x in X if for every sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(x_n, x, z) = 0$ for all $z \in X$, $\lim_{n\to\infty} d(Sx_n, Sx, z)=0$.

Note that, in a 2-metric space (X, d), a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X ([18]).

DEFINITION 2.5. Let S and T be mappings from a 2-metric space (X, d) into itself. S and T are said to be *weakly commuting* if

$$d(STx, TSx, z) \leq d(Sx, Tx, z)$$

for all $x, z \in X$.

DEFINITION 2.6. Let S and T be mappings from a 2-metric space (X, d) into itself. S and T are said to be *compatible* if

$$\lim_{n\to\infty} d(STx_n, TSx_n, z) = 0$$

for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X.

Note that any commuting mappings are weakly commuting but the converse is not true ([18]). In turn, any weakly commuting mappings are compatible but the converse is not ture ([9]).

DEFINITION 2.7. Let S and T be mappings from a 2-metric space (X, d) into itself. S and T are said to be *compatible of type* (A) if

$$\lim_{n\to\infty} d(TSx_n, SSx_n, z) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(STx_n, TTx_n, z) = 0$$

for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

The following propositions show that Definitions 2.6 and 2.7 are equivalent under some conditions ([17]) :

PROPOSITION 2.1. *Let S and T be sequentially continuous mappings from a 2-metric space (X, d) into itself. If S and T are compatible, then they are compatible of type* (A)

PROPOSITION 2.2. *Let S and T be compatible mappings of type* (A) *from a 2-metric space (X, d) into itself. If one of S and T is sequentially continuous, then S and T are compatible.*

As a direct consequence of Propositions 2.1 and 2.2, we have the following :

PROPOSITION 2.3. *Let S and T be sequentially continuous mappings from a 2-metric space (X, d) into itself. Then S and T are compatible if and only if they are compatible of type* (A).

In [10], G.Jungck et al. gave two examples to show that Proposition 2.3 is not ture when two mappings S and T are not continuous. Next, we give some properties of compatible mappings of type (A) for our main theorems ([17]) :

PROPOSITION 2.4. *Let S and T be compatible mappings of type (A) from a 2-metric space (X, d) into itself. If $St = Tt$ for some $t \in X$, then $STt = TTt = TSt = SSt$.*

PROPOSITION 2.5. *Let S and T be compatible mappings of type (A) from a 2-metric space (X, d) into itself. Suppose that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X. Then we have the following :*

(1) $\lim_{n\to\infty} TSx_n = St$ if S is sequentially continuous at t.

(2) $STt = TSt$ and $St = Tt$ if S and T are sequentially continuous at t.

3. Some lemmas

In this section, we introduce some lemmas for our main theorems.

Let A, B, S and T be mappings from a 2-metric space (X, d) into itself satisfying the following conditions :
(3.1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,
(3.2) there exists an $h \in (0.1)$ such that

$$d(Ax, By, z) \leq h \max\{d(Sx, Ty, z), d(Ax, Sx, z), d(By, Ty, z), \frac{1}{2}(d(Ax, Ty, z) + d(By, Sx, z))\}$$

for all $x, y, z \in X$.
Then, by (3.1), since $A(X) \subseteq T(X)$, for an arbitray point $x_o \in X$, there exists a point $x_1 \in X$ such that $Ax_o = Tx_1$. Since $B(X) \subseteq S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can difine a sequence $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \tag{3.3}$$

for $n = 0, 1, 2, \ldots$.

LEMMA 3.1. *Let A, B, S and T be mappings from a 2-metric spcae (X, d) into itself satisfying the conditions (3.1) and (3.2). Then we have $d(y_i, y_j, y_k) = 0$ for $i, j, k = 0, 1, 2, \ldots$, where $\{y_n\}$ is the sequence defined by (3.3).*

Proof. In (3.2), taking $x = x_{2n+2}, y = x_{2n+1}$ and $z = y_{2n}$, we have

$$
\begin{aligned}
&d(Ax_{2n+2}, Bx_{2n+1}, y_{2n}) \\
&\quad \le h \max\{d(Sx_{2n+2}, Tx_{2n+1}, y_{2n}), \\
&\qquad d(Ax_{2n+2}, Sx_{2n+2}, y_{2n}), d(Bx_{2n+1}, Tx_{2n+1}, y_{2n}), \\
&\qquad \frac{1}{2}(d(Ax_{2n+2}, Tx_{2n+1}, y_{2n}) + d(Bx_{2n+1}, Sx_{2n+2}, y_{2n}))\}
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
d(y_{2n+2}, y_{2n+1}, y_{2n}) &\le h \max\{d(y_{2n+1}, y_{2n}, y_{2n}), \\
&\qquad d(y_{2n+2}, y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n}, y_{2n}), \\
&\qquad \frac{1}{2}(d(y_{2n+2}, y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n+1}, y_{2n}))\} \\
&= h \max\{0, d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, 0\} \\
&= h\ d(y_{2n+2}, y_{2n+1}, y_{2n}),
\end{aligned}
$$

which is a contracdiction. Thus, we have $d(y_{2n+2}, y_{2n+1}, y_{2n}) = 0$. Similarly, we have $d(y_{2n+1}, y_{2n+2}, y_{2n+3}) = 0$. Hence, for $n = 0, 1, 2, \dots$, it follows that

$$
d(y_n, y_{n+1}, y_{n+2}) = 0. \tag{3.4}
$$

Next, for all $z \in X$, let $d_n(z) = d(y_n, y_{n+1}, z)$, $n = 0, 1, 2, \dots$. By (3.4), we have

$$
\begin{aligned}
d(y_n, y_{n+2}, z) &\le d(y_n, y_{n+2}, y_{n+1}) + d(y_n, y_{n+1}, z) + d(y_{n+1}, y_{n+2}, z) \\
&= d(y_n, y_{n+1}, z) + d(y_{n+1}, y_{n+2}, z) \\
&= d_n(z) + d_{n+1}(z).
\end{aligned}
$$

Taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (3.2), we have

$$
\begin{aligned}
d_{2n+1}(z) &= d(y_{2n+2}, y_{2n+1}, z) \\
&= d(Ax_{2n+2}, Bx_{2n+1}, z) \\
&\leq h \max\{d(Sx_{2n+2}, Tx_{2n+1}, z), \\
&\qquad d(Ax_{2n+2}, Sx_{2n+2}, z), d(Bx_{2n+1}, Tx_{2n+1}, z), \\
&\qquad \frac{1}{2}(d(Ax_{2n+2}, Tx_{2n+1}, z) + d(B_{2n+1}, Sx_{2n+2}, z))\} \\
&= h \max\{d(y_{2n+1}, y_{2n}, z), \qquad\qquad (3.5) \\
&\qquad d(y_{2n+2}, y_{2n+1}, z), d(y_{2n+1}, y_{2n}, z), \\
&\qquad \frac{1}{2}(d(y_{2n+2}, y_{2n}, z) + d(y_{2n+1}, y_{2n+1}, z))\} \\
&\leq h \max\{d_{2n}(z), d_{2n+1}(z), d_{2n}(z), \\
&\qquad \frac{1}{2}(d_{2n}(z) + d_{2n+1}(z))\}.
\end{aligned}
$$

Now, we shall show that $\{d_n(z)\}$ is a non-increasing sequence in R^+. In fact, by (3.2), we have

$$d_{2n}(z) \leq h\ d_{2n-1}(z) \leq d_{2n-1}(z)$$

for every integer n. Now, suppose that $d_{n+1}(z) > d_n(z)$ for some n. By (3.5), we have $d_{2n+1}(z) \leq hd_{2n+1}(z)$ for some $h \in (0.1)$, which is a contradiction, since $d_{2n+1}(z) > 0$. Therefore, the sequence $\{d_n(z)\}$ is non-increasing in R^+.

Now, we claim that $d_n(y_m) = 0$ for all non-negative integers m, n.

Case 1. $n \geq m$. Then we have $0 = d_m(y_m) \geq d_n(y_m)$.

Case 2. $n < m$. By (M_4), we have

$$
\begin{aligned}
d_n(y_m) &\leq d_n(y_{m-1}) + d_{m-1}(y_n) \\
&\leq d_n(y_{m-1}) + d_n(y_n) \\
&= d_n(y_{m-1}).
\end{aligned}
$$

By using the above inequality repeatedly, we have

$$d_n(y_m) \leq d_n(y_{m-1}) \leq \cdots \leq d_n(y_n) = 0,$$

which completes the proof of our claim.

Finally, let i, j and k be arbitrary non-negative integers. We may assume that $i < j$. By (M_4), we have

$$\begin{aligned}(y_i, y_j, y_k) &\leq d_j(y_i) + d_j(y_k) + d(y_{i+1}, y_j, y_k)\\ &= d(y_{i+1}, y_j, y_k).\end{aligned}$$

Therefore, by repeatition of the above inequality, we have

$$d(y_i, y_j, y_k) \leq d(y_{i+1}, y_j, y_k) \leq \cdots \leq d(y_i, y_j, y_k) = 0.$$

This completes the proof.

LEMMA 3.2. *Let A, B, S and T be mappings from a 2-metric space (X, d) into itself satisfying the conditions* (3.1) *and* (3.2). *Then the sequence $\{y_n\}$ defined by* (3.3) *is a Cauchy sequence in X.*

Proof. In the proof of Lemma 3.1, since $\{d_n(z)\}$ is a non-increasing sequence in R^+, by (3.2), we have

$$\begin{aligned}d_1(z) &= d(y_1, y_2, z)\\ &= d(Ax_2, Bx_1, z)\\ &\leq h \max\{d(Sx_2, Tx_1, z), d(Ax_2, Sx_2, z), d(Bx_1, Tx_1, z),\\ &\qquad \frac{1}{2}(d(Ax_2, Tx_1, z) + d(Bx_1, Sx_2, z))\}\\ &= h \max\{d(y_1, y_0, z), d(y_2, y_1, z), d(y_1, y_0, z),\\ &\qquad \frac{1}{2}(d(y_2, y_0, z) + d(y_1, y_1, z))\}\\ &\leq h\ d_0(z).\end{aligned}$$

In general, we have $d_n(z) \leq h^n d_0(z)$, which implies that $\lim_{n\to\infty} d_n(z) = 0$. Now, we shall prove that $\{y_n\}$ is a Cauchy sequence in X. Since $\lim_{n\to\infty} d_n(z) = 0$, it is sufficient to show that a subsequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in X. Suppose that the sequence $\{y_{2n}\}$ is not a Cauchy sequence in X. Then there exist an $\epsilon > 0$ and strictly increasing sequences $\{m_k\}, \{n_k\}$ of positive intergers such that $k \leq n_k < m_k$,

$$(3.6) \qquad d(y_{2n_k}, y_{2m_k}, z) \geq \epsilon \quad \text{and} \quad d(y_{2n_k}, y_{2m_k-2}, z) < \epsilon$$

for all $k = 1, 2, \ldots$. By Lemma 3.1 and (M_4), we have

$$d(y_{2n_k}, y_{2m_k}, z) - d(y_{2n_k}, y_{2m_k-2}, z) \le d(y_{2m_k-2}, y_{2m_k}, z) \le d_{2m_k-2}(z) + d_{2m_k-1}(z).$$

Since $\{d(y_{2n_k}, y_{2m_k}, z) - \epsilon\}$ and $\{\epsilon - d(y_{2n_k}, y_{2m_k-2}, z)\}$ are sequences in R^+ and $\lim_{n\to\infty} d_n(z) = 0$, we have

$$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k}, z) = \epsilon \quad \text{and} \quad \lim_{k\to\infty} d(y_{2n_k}, y_{2m_k-2}, z) = \epsilon. \tag{3.7}$$

Note that, by (M_4),

$$|d(x, y, a) - d(x, y, b)| \le d(a, b, x) + d(a, b, y) \tag{3.8}$$

for all $x, y, a, b \in X$. Taking $x = y_{2n_k}, y = a, a = y_{2m_k-1}$ and $b = y_{2m_k}$ in (3.8) and using Lemma 3.1 and (3.7), we have

$$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k-1}, z) = \epsilon. \tag{3.9}$$

Once again, by using Lemma 3.1, (3.7) and (3.8), we have

$$\lim_{k\to\infty} d(y_{2n_k+1}, y_{2m_k}, z) = \epsilon \quad \text{and} \quad \lim_{k\to\infty} d(y_{2n_k-1}, y_{2m_k-1}, z) = \epsilon.$$

Thus, by (3.2), we have

$$\begin{aligned} d(y_{2m_k}, y_{2n_k+1}, z) &= d(Ax_{2m_k}, Bx_{2n_k+1}, z) \\ &\le h \max\{d(Sx_{2m_k}, Tx_{2n_k+1}, z), \\ &\quad d(Ax_{2m_k}, Sx_{2m_k}, z), d(Bx_{2n_k+1}, Tx_{2n_k+1}, z), \\ &\quad \frac{1}{2}(d(Ax_{2m_k}, Tx_{2n_k+1}, z) + d(Bx_{2n_k+1}, Sx_{2m_k}, z))\} \\ &= h \max\{d(y_{2m_k-1}, y_{2n_k}, z), \\ &\quad d(y_{2m_k}, y_{2m_k-1}, z), d(y_{2n_k+1}, y_{2n_k}, z), \\ &\quad \frac{1}{2}(d(y_{2m_k}, y_{2n_k}, z) + d(y_{2n_k+1}, y_{2m_k-1}, z))\}. \end{aligned} \tag{3.10}$$

Letting $k \to \infty$ in (3.10) and noting that d is continuous, we have $\epsilon \le h\ \epsilon < \epsilon$, which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in X and so the sequence $\{y_n\}$ defined by (3.3) is a Cauchy sequence in X. This completes the proof.

The following lemma was introduced by S.L. Singh ([22], [23]) :

LEMMA 3.3. *Let $\{x_n\}$ be a sequence in a complete 2-metric space (X, d). If there exists an $h \in (0,1)$ such that*

$$d(x_n, x_{n+1}, z) \le hd(x_{n-1}, x_n, z)$$

for all $z \in X$ and $n = 1, 2, \dots$, then the sequence $\{x_n\}$ is a Cauchy sequence in X.

4. Common fixed point theorems

In this section, we prove two kinds of common fixed point theorems for compatible mappings of type (A) in 2-metric spaces. First, by using Lemma 3.2, we prove the following :

THEOREM 4.1. *Let A, B, S and T be mappings from a complet 2-metric space (X, d) into itself satisfying the conditions* (3.1), (3.2), (4.1) *and* (4.2) :

(4.1) *one of A, B, S and T is sequentially continuous,*

(4.2) *the pairs A, S and B, T are compatible mappings of type (A).*

Then A, B, S and T have a unique common fixed point in X.

Proof. By Lemma 3.2, since the sequence $\{y_n\}$ defined by (3.3) is a Cauchy sequence in X and (X, d) is complete, it converges to a point u in X and so the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ of $\{y_n\}$ also converge to the point u, that is, $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n+2}\}$ and $\{Tx_{2n+1}\}$ converge to u. Now, suppose that T is sequentially continuous. Since B and T are compatible mappings of type (A), by Proposition 2.5, we have $BTx_{2n+1}, TTx_{2n+1} \to Tu$ as $n \to \infty$. Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in (3.2), we have

$$\begin{aligned} &d(Ax_{2n}, BTx_{2n+1}, z) \\ &\quad \le h \max\{d(Sx_{2n}, TTx_{2n+1}, z), \\ &\qquad d(Ax_{2n}, Sx_{2n}, z), d(BTx_{2n+1}, TTx_{2n+1}, z), \\ &\qquad \frac{1}{2}(d(Ax_{2n}, TTx_{2n+1}, z) + d(BTx_{2n+1}, Sx_{2n}, z))\}. \end{aligned} \tag{4.3}$$

As $n \to \infty$ in (4.3), we have

$$\begin{aligned} d(u, Tu, z) &\le h \max\{d(u, Tu, z), d(u, u, z), d(Tu, Tu, z), \\ &\qquad \frac{1}{2}(d(u, Tu, z) + d(Tu, u, z))\} \\ &\le h\ d(u, Tu, z), \end{aligned}$$

which implies $Tu = u$. Again, replacing x by x_{2n} and y by u in (3.2), respectively, we have

$$d(Ax_{2n}, Bu, z) \le h \max\{d(Sx_{2n}, Tu, z), d(Ax_{2n}, Sx_{2n}, z), \\ d(Bu, Tu, z), \frac{1}{2}(d(Ax_{2n}, Tu, z) + d(Bu, Sx_{2n}, z))\}.$$

As $n \to \infty$ in (4.4), we have

$$\begin{aligned} d(u, Bu, z) &\le h \max\{d(u, Tu, z), d(u, u, z), d(Bu, u, z), \\ &\qquad \frac{1}{2}(d(u, Tu, z) + d(Bu, u, z))\} \\ &\le h\ d(u, Bu, z), \end{aligned}$$

which implies $Bu = u$. Since $B(X) \subsetneq S(X)$, there exists a point $w \in X$ such that $Bu = Sw = u$. By using (3.2) again, we have

$$\begin{aligned} d(Aw, u, z) &= d(Aw, Bu, z) \\ &\le h \max\{d(Sw, Tu, z), d(Aw, Sw, z), d(Bu, Tu, z), \\ &\qquad \frac{1}{2}(d(Aw, Tu, z) + d(Bu, Sw, z))\} \\ &\le h\ d(Aw, u, z), \end{aligned}$$

which implies $Aw = u$. But since A and S are compatible mappings of type (A) and $Aw = Sw = u$, by Proposition 2.4, $d(ASw, SSw, z) = 0$ and hence we have $Au = ASw = SSw = Su$. From (3.2), we have also

$$\begin{aligned} d(Au, u, z) &= d(Au, Bu, z) \\ &\le h \max\{d(Su, Tu, z), d(Au, Su, z), d(Bu, Tu, z), \\ &\qquad \frac{1}{2}(d(Au, Tu, z) + d(Bu, Su, z))\} \\ &\le h\ d(Au, u, z), \end{aligned}$$

which implies $Au = u$. Therefore, we $Au = Bu = Su = Tu = u$, that is, u is a common fixed point of A, B, S and T. The uniqueness of the point u follows easily from (3.2). Similarly, we can also complete the proof when A or B or S is sequentially continuous. This completes the proof.

Putting $A = B$ and $S = T$ in Theorem 4.1, we have the following :

COROLLARY 4.2. *Let A and S be mappings from complete a 2-metric space (X, d) into itself satisfying the following conditions :*

(4.5) *one of A, S is sequentially continuous,*

(4.6) $A(X) \subseteq S(X)$,

(4.7) *A and S are compatible mappings of type* (A),

(4.8) *there exists an $h \in (0, 1)$ such that*

$$d(Ax, Ay, z) \leq h \max\{d(Sx, Sy, z), d(Ax, Sx, z), d(Ay, Sy, z)$$
$$\frac{1}{2}(d(Ax, Sy, z) + d(Ay, Sx, z))\}$$

for all $x, y, z \in X$.

Then A and S have a unique common fixed point in X.

Secondly, by using Lemma 3.3, we prove the following :

THEOREM 4.3. *Let A, B, S and T be mappings from a complete 2-metric space (X, d) into itself satisfying the conditions* (3.2), (4.2), (4.9), (4.10) *and* (4.11) :

(4.9) $AT(X) \cup BS(X) \subseteq ST(X)$,

(4.10) $ST = TS$,

(4.11) *S and T are sequentially continuous.*

Then (1) *A and S have a common fixed point u in X,* (2) *B and T have a common fixed point u in X. Indeed, the point u is a unique common fixed point of A, B, S and T.*

Proof Since $AT(X) \cup BS(X) \subseteq ST(X)$, for an arbitrary point $x_o \in X$, we can construct a sequence $\{x_n\}$ in X such that

$$ATx_{2n} = STx_{2n+1} \quad \text{and} \quad BSx_{2n+1} = STx_{2n+2}$$

for $n = 0, 1, 2, \ldots$. By (3.2), we have

$$\begin{aligned}
&d(STx_{2n+1}, ST_{2n+2}, z) \\
&\quad = d(ATx_{2n}, BSx_{2n+1}, z) \\
&\quad \leq h \max\{d(STx_{2n}, TSx_{2n+1}, z), \\
&\qquad d(ATx_{2n}, STx_{2n}, z), d(BSx_{2n+1}, TSx_{2n+1}, z) \\
&\qquad \frac{1}{2}(d(ATx_{2n}, TSx_{2n+1}, z) + d(BSx_{2n+1}, STx_{2n}, z))\} \\
&\quad = h \max\{d(STx_{2n}, STx_{2n+1}, z), \\
&\qquad d(STx_{2n+1}, STx_{2n}, z), d(STx_{2n+2}, STx_{2n+1}, z), \\
&\qquad \frac{1}{2}(d(STx_{2n+1}, STx_{2n+1}, z) + d(STx_{2n+2}, STx_{2n}, z))\} \\
&\quad = h \max\{d(STx_{2n}, STx_{2n+1}, z), \frac{1}{2} d(STx_{2n+2}, STx_{2n}, z)\}
\end{aligned} \tag{4.12}$$

and so we have $d(STx_{2n+1}, STx_{2n+2}, STx_{2n}) = 0$. From (4.12), we have

$$d(STx_{2n+1}, STx_{2n+2}, z) \leq h\, d(STx_{2n}, STx_{2n+1}, z).$$

Similarly, we have

$$d(STx_{2n+2}, STx_{2n+3}, z) \leq h\, d(STx_{2n+1}, STx_{2n+2}, z).$$

Thus, for $n = 0, 1, 2, \ldots$, we have

$$d(STx_{n+1}, STx_{n+2}, z) \leq h\, d(STx_n, STx_{n+1}, z)$$

for all $z \in X$. Therefore, by Lemma 3.3, the sequence $\{STx_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, it converges to a point $u \in X$ and so the subsequences $\{ATx_{2n}\}$ and $\{BSx_{2n+1}\}$ of $\{STx_n\}$ also converge to the point u.

Now suppose that S is sequentially continuous. Then we have $SSTx_n \to Su$ as $n \to \infty$. Since $\{ATx_{2n}\}$ and $\{STx_{2n}\}$ converge to the point u, by Proposition 2.4, $ASTx_{2n} \to Su$ as $n \to \infty$. Thus, putting $x = STx_{2n}$ and $y = Sx_{2n+1}$ in (3.2), we have

$$\begin{aligned}
&d(ASTx_{2n}, BSx_{2n+1}, z) \\
&\quad \leq h \max\{d(SSTx_{2n}, TSx_{2n+1}, z), \\
&\qquad d(ASTx_{2n}, SSTx_{2n}, z), d(BSx_{2n+1}, TSx_{2n+1}, z), \\
&\qquad \frac{1}{2}(d(ASTx_{2n}, TSx_{2n+1}, z) + d(BSx_{2n+1}, SSTx_{2n}, z))\}.
\end{aligned} \tag{4.13}$$

As $n \to \infty$ in (4.13), we have $d(Su, u, z) \leq h\ d(Su, u, z)$ and so, since $h \in (0, 1)$, we have $Su = u$. Again, putting $x = u$ and $y = Sx_{2n+1}$ in (3.2), we have

$$
\begin{aligned}
& d(Au, BSx_{2n+1}, z) \\
& \quad \leq h \max\{d(Su, TSx_{2n+1}, z), \\
(4.14) \qquad & \qquad d(Au, Su, z), d(BSx_{2n+1}, TSx_{2n+1}, z), \\
& \qquad \frac{1}{2}(d(Au, TSx_{2n+1}, z) + d(BSx_{2n+1}, Su, z))\}.
\end{aligned}
$$

As $n \to \infty$ in (4.14), we have also $d(Au, u, z) \leq h\ d(Au, u, z)$ and so $Au = u$. Therefore, $Au = Su = u$, that is, u is a common fixed point of A and S. Similarly, in the case that T is sequentially continuous and B, T are compatible mappings of type (A), we have $Bu = Tu = u$, that is, u is a common fixed point of B and T.

Finally, we shall prove that the point u is a unique common fixed point of A, B, S and T. In fact, by using (3.2), we have

$$
\begin{aligned}
d(Au, Bu, z) \leq h \max\{& d(Su, Tu, z), d(Au, Su, z), d(Bu, Tu, z), \\
& \frac{1}{2}(d(Au, Tu, z) + d(Bu, Su, z))\},
\end{aligned}
$$

which implies $Au = Bu$. Therefore, u is a common fixed point of A, B, S and T, and the uniqueness of the point u follows easily from (3.2). This completes the proof.

REMARK. (1) Theorems 4.1 and 4.3 extend, generalize and improve some fixed point theorems for commuting mappings and weakly commuting mappings in 2-metric spaces ([8], [13], [15], [19], [20], [22]-[24]).

(2) If all mappings in Theorems 4.1 and 4.3 are sequentially continuous, then Theorems 4.1 and 4.3 are still true even though the condition (4.3) is replaced by the compatibility from Proposition 2 3.

(3) If we put $A = B$ and $S = T$ in Theorem 4.3, then we have also Corollary 4.2.

References

1. Y J Cho, M S Khan and S.L. Singh,, *Common fixed points of weakly commuting mappings*,, Review of Research, Faculty of Science, Math. Series, Univ. of Novi Sad, **18** (1988), 129–142

2. A. Constantin, *Common fixed points of weakly commuting mappings in 2-metric spaces*, Math. Japon. **36** (1991), 507–514.
3. S. Gähler, *2-metrische Räume und ihr topologische Struktur*, Math. Nachr. **26** (1963), 115–148.
4. S. Gähler, *Über dei Uniformisierbarkeit 2-metrische Räume*, Math. Nachr. **28** (1965), 235–244.
5. S. Gähler, *Zur Geometrie 2-metrische Räume*, Rev. Roum. De Math. Pures et Appl. **11** (1966), 665–667.
6. M. Imdad, M.S. Khan and M.D. Khan, *A common fixed point theorem in 2-metric spaces*, Math. Japon. **36** (1991), 907–914.
7. K. Iseki, *A Property of orbitally continuous mappings on 2-metric spaces*, Math. Seminar Notes, Kobe Univ. **3** (1975), 131–132.
8. K. Iseki, P.L. Sharma and B.K. Sharma, *Contraction type mappings on 2-metric spaces*, Math. Japon. **21** (1978), 67–70.
9. G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. and Math Sci. **9** (1986), 771–779
10. G. Jungck, P.P. Murthy and Y.J Cho, *Compatible mappings of type (A)*, to appear in Math Japon.
11. M.S. Khan, *Convergence of sequences of fixed points in 2-metric spaces*, Indian J Pure and Appl. Math. **10** (1979), 1062–1067.
12. M.S. Khan On fixed point theorems in 2-metric spaces, Publ. Inst. Math. (Beograd) (N.S.) **41** (1980), 107–112.
13. M.S. Khan and B Fisher, *Some fixed point theorems for commuting mappings*, Math. Nachr. **106** (1982), 323–326.
14. M.S. Khan, M Imdad and M. Swaleh, *Asymptotically regular maps and sequences in 2-metric spaces*, Indian J. Math **27** (1985), 81–88.
15. T. Kubiak, *Common fixed points of pairwise commuting mappings*, Math. Nachr. **118** (1984), 123–127
16 S N. Lal and A.K. Singh, *An analogue of Banach's contraction principle for 2-metric spaces*, Bull. Austral. Math. Soc. **18** (1978), 137–143.
17. P.P. Murthy, S.S. Chang, Y.J Cho and B.K. Sharma, *Compatible mappings of type (A) and common fixed point theorems*, to appear in Kyungpook Math. J.
18. S.V.R. Naidu and J R. Prasad, *Fixed point theorems in 2-metric spaces*, Indian J. Pure and Appl Math **17** (1986), 974–993.
19. B. Ram, *Existence of fixed points in 2-metric spaces*, Ph.D. Thesis, Garhwal Univ. (1982).
20. B.E. Rhoades, *Contraction type mappings on a 2-metric space*, Math Nachr **91** (1979), 151–154.
21 A.K. Sharma, *On fixed points in 2-metric spaces*, Math Seminar Notes, Kobe Univ **6** (1978), 467–473
22. S.L. Singh, *Some contraction type principles on 2-metric spaces and applications*, Math. Seminar Notes, Kobe Univ. **7** (1979), 1–11.
23 S.L. Singh, B M.L Tiwari and V.K. Gupta, *Common fixed points of commuting mappings in 2-metric spaces and application*, Math. Nachr. **95** (1980), 293–297.

24. S.L. Singh and B. Ram, *A note on the convergence of sequence of mappings and their common fixed points in a 2-metric space*, Math. Seminar Notes, Kobe Univ. 9 (1981), 181–185.

Department of Mathematics
Gyeongsang National University
Jinju 660-701, Korea

Institute of Mathematics
A. Mickiewicz University
Matejki 48/49
60-769 Poznan, Poland
and
Department of Mathematics
Rhodes University
P.O. Box 94
Grahamstown 6140, South Africa