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SELF MAPPINGS WITH CLOSED GRAPHS

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1. Introduction

Let X be a topological space and $f : X \to X$ the self mapping. f is said to have its closed graph iff the set $G_f = \{(x, f(x)) : x \in X\}$ is closed in $X \times X$. Equivalently, there exist a neighbohood (written by nbd) U of x and a nbd V of y such that $f(U) \cap V = \emptyset$ whenever $(x, y) \notin G_f$. In the articles [4], [5] and [6], implications of mappings with closed graphs have been examined. Separately in [1], several situations have been investigated regarding the composition of mappings with closed graphs.

This note is in fact an outcome of our basic attempt to give an unified approach for separation axioms. However, we have been partially able to obtain such properties induced by the family of self mappings under some suitable conditions.

In the next section, we have also studied some algebraic properties of such family in the sense of their composition. Conversely, some results regarding each component of the composition have also been included. At the end, a result on the closed graph property of the evaluation mapping has been derived.

2. Notations and terminologies

Let X be the given topological space. Then

(1) $F(X)$ denotes the family of all self mappings on X.

(2) $G(X)$ denotes the family of all self mappings on X with closed graphs.

(3) $CG(X)$ denotes the family of all self continuous mappings on X with closed graphs.

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(4) $C(X)$ denotes the family of all self continuous mappings on X. For X being T_2-space, $C(X) \subset G(X)$ ([3]). In general, we shall be dealing with non Hausdorff spaces unless the same is specified.

(5) We say that $G(X)$ or $CG(X)$ is a separating family if for $x \neq y$ in X there is $f \in G(X)$ or $f \in CG(X)$ such that $f(x) \neq f(y)$.

Remaining terms and terminologies have been drawn from [2] and [3].

PROPOSITION 2.1. *X is T_1-space if $G(X)$ is a separating family.*

Proof. Let $x \neq y$ in X. Then there is $f \in G(X)$ such that $f(x) \neq f(y)$. From [6] $f(x) \notin \overline{f(V_y)}$ for at least a nbd V_y of y, i.e., there is a nbd $V_{f(x)}$ of $f(x)$ such that

$$V_{f(x)} \cap f(V_y) = \emptyset \Longrightarrow f^{-1}(V_{f(x)}) \cap V_y = \emptyset \Longrightarrow x \notin V_y.$$

Similarly, there exists a nbd V_x of x such that $y \notin V_x$, i.e., X is T_1-space.

PROPOSITION 2.2. *X is T_2-space if $CG(X)$ is the separating family.*

Proof. For $x \neq y$ in X there is $f \in CG(X)$ such that $f(x) \neq f(y)$. As done in the earlier proposition there exist nbds $V_{f(x)}$ and V_y of $f(x)$ and y respectively such that $V_{f(x)} \cap f(V_y) = \emptyset$. Again f is continuous also so there is a nbd U_x of x such that $f(U_x) \subset V_{f(x)}$ and hence $f(U_x) \cap f(V_y) = \emptyset$. This implies that $U_x \cap V_y = \emptyset$, i.e., x and y have disjoint nbds, i.e., X is T_2-space.

REMARK 2.3. From Proposition 2.2 constant mappings have closed graph snd they are continuous and hence $CG(X)$ or $G(X) = \emptyset$. For non Hausdorff spaces, the separating family $CG(X)$ may be empty in view of the example given below.

EXAMPLE 2.4. Let X be an infinite set equipped with cofinite topology. Then the separating family $CG(X) = \emptyset$, for if not so let $f \in CG(X)$ such that $f(y) \neq f(x)$ for $x \neq y$ in X. Then there exist nbds $V_{f(x)}$ of $f(x)$ and V_y of y such that $V_{f(x)} \cap f(V_y) = \emptyset$ or $f^{-1}(V_{f(x)}) \cap V_y = \emptyset$. Topology is cofinite so setting $f^{-1}(V_{f(x)}) = X - F_1$ and $V_y = X - F_2$ where F_1 and F_2 are finite subsets of X, we have $(X - F_1) \cap (X - F_2) = \emptyset$. This implies $F_1 \cup F_2 = X$ which is a contradiction for X is infinite and hence the separating family $CG(X) = \emptyset$.

3. Algebra of $G(X)$

Composition of self mappings need not have its graph closed even if one of the components has its graph closed. Again, if the composition has closed graph, it is not implied that its components have closed graph.

EXAMPLE 3.1. Let $X = R$ be the set of real numbers equipped with usual topology. Define $f : R \to R$ as

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational,} \end{cases}$$

and $g : R \to R$ as

$$g(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

$g \circ f$ has closed graph where as its component f is neither continuous nor has its graph closed and the other component g has closed graph but not continuous.

EXAMPLE 3.2. Define $f : R \to R$ as considered in the above example and let I be the identity mapping. $f \circ I$ has not its graph closed where as I being continuous with closed graph and the other component f neither continuous nor with closed graph.

Similar such investigation led us to the following results.

PROPOSITION 3.3. *Let $f \in C(X)$ and $g \in G(X)$. Then $g \circ f \in G(X)$.*

Proof. Let $y \neq g \circ f(x)$ for any pair $(x, y) \in X \times X$, i.e., $y \neq g(f(x))$. g has closed graph and so there exist a nbd V_y of y and a nbd $V_{f(x)}$ of $f(x)$ such that $V_y \cap g(V_{f(x)}) = \emptyset$. Again f is continuous so there is a nbd U_x of x such that $f(U_x) \subset V_{f(x)}$ and hence $V_y \cap g(f(U_x)) = \emptyset$, i.e., graph of $g \circ f$ is closed and hence $g \circ f \in G(X)$.

Conversely, if the composition of self mappings has closed graph, then we have the following propositions.

PROPOSITION 3.4. *Let $g \circ f \in G(X)$. Then $f \in G(X)$ whenever g is a continuous injection.*

Proof. For each pair $(x, y) \notin G_f$, $y \neq f(x) \Longrightarrow g(f(x)) \neq g(y)$ for g is injection, i.e., $g \circ f(x) \neq g(y)$. $g \circ f$ has closed graph so there exist a nbd W containing $g(y)$ and a nbd U containing x such that $g \circ f(U) \cap W = \emptyset$. So $f(U) \cap g^{-1}(W) = \emptyset$. Setting $V = g^{-1}(W)$ which is open for g is continuous, we have

$$f(U) \cap V = \emptyset \Longleftrightarrow (U \times V) \cap G_f = \emptyset,$$

i.e., $(x, y) \notin \overline{G_f}$ and hence $G_f = \overline{G_f}$, i.e., graph of f is closed.

Analogously, it is easy to prove the following proposition.

PROPOSITION 3.5. *If $g \circ f \in G(X)$, then $g \in G(X)$ whenever f is an open surjection.*

COROLLARY. *If f and g are homeomorphism such that $g \circ f \in G(X)$, then f and g both belong to $G(X)$.*

We close this article just by proving a result concerning the closed graph property in the context of the evaluation mapping. For a topological space X, the evaluation mapping $e : X \to \Pi\{X_f : f \in F(X)\}$ where each $X_f = X$ defined as $< e(x) >_f = f(x)$ for each $x \in X$.

PROPOSITION 3.6. *The evaluation mapping e has closed graph if $F(X) = G(X)$.*

Proof. To this end, for each pair $e(x) \neq < y_f >$, i.e., they must differ in at least one coordinate space say the f-th coordinate space. i.e., $< e(x) >_f \neq y_f$. Hence $f(x) \neq y_f$. Since $f \in G(X)$, there exist nbds U and V containing x and y_f, respectively, such that

$$f(U) \cap V = \emptyset \Longrightarrow [e(U)]_f \cap V = \emptyset.$$

Setting $\Pi\{V_g : g \in F(X)\}$ where $V_g = V$ if $g = f$, otherwise $V_g = X$, we have $e(U) \cap \Pi\{V_g : g \in F(X)\} = \emptyset$, i.e., graph of the evaluation mapping e is closed.

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References

1. G.I. Chae, V.P. Singh and D N. Misra, *Composition of functions with closed graphs and its function spaces*, Pusan Kyŏngnam Math. J. **5** (1989), 151–154.
2. J.L. Kelley, *General Topology*, Van Nostrand, New York, 1955
3. J. Dugundji, *Topology*, Allyn and Bacon, 1965
4. P.E. Long, *Functions with closed graph*, Amer. Math. Monthly **76** (1969), 930–932.
5. V.P. Singh and D.N. Misra, *Some implications of mappings with closed graphs*, Proc. International Symp. Modern Analysis and its Applications published by Prentice Hall of India Pvt. New Delhi (1986), 13–15.
6. V.P. Singh, G.I Chae and D N. Misra, *Functions with closed graph and some other related results*, UOU Report **17** (1986), 127–131.

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