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UNIFORM DIMESION OF FINITE SUBNORMALIZING EXTENSIONS

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1. Introduction

In 1986, E.A. Whelan ([2]) studied some properties of more generalized ring extensions which are finite subnormaizing extensions and self conjugate extensions. He ggave some examples of those extensions which are not finite normalizing extensions and showed that the cutting down property is also true in finite subnormalizing extensions.

In this paper we will show that if any S-module M has finite uniform dimension if and only if it has finite uniform dimension when considered as an R-module where S is a finite subnormalizing extension of a ring R. And we will give some counterexample which show some properties that are not true in finite subnormalizign extensions but are true in finite normalizing extensions.

DEFINITION 1. Let M be an $R-S$ bimodule where R and S are any rings.

(1) x is called normalizing in M where $x \in M$ if $Rx = xS$.

(2) $X \subset M$ is called self conjugate in M if $RX = XS$.

(3) A sequence $\{x_1, x_2, \cdots, x_n, \cdots\}$ in M is called a subnormalizing sequence if $X_j = \{x_1, x_2, \cdots, x_j\}$ is self conjuage for every j.

From this definition we get the following definition.

DEFINITION 2. Let R be a subring of a ring S and there exists some finite subset $X \subset S$ such that $S = RX$ that is $S = \Sigma Rx_i$ where $X = \{x_1, x_2, \cdots, x_n\}$.

(1) S is called a finite normalizing extension of a ring R if $Rx_i = x_iR$ for every i.

(2) S is called a finite subnormalizing extension of a ring R if X is a subnormalizing sequence.

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(3) S is called a finite self conjugate extension of a ring R if X is self conjugate in S.

The following example show that there exist finite subnormalizing extension which are not finite normalizing extension.

EXAMPLE 3. Let K be a field and R be the set of all upper triangular matrices and S be the set of all 2×2 matrices over K. Then S is a finite subnormalizing extension of R but not a finite normalizing extension of R.

EXAMPLE 4. Let K be an extension field of a field F such that $\dim_F K = n \geq 2$. Let $R = \begin{pmatrix} F & 0 \\ 0 & K \end{pmatrix}$ and $S = \begin{pmatrix} F & K \\ 0 & K \end{pmatrix}$. Then S is self conjugate extension but not a finite subnormalizing extension of R.

2. Main theorem

In this paper we assume that S is a finite subnnormalizing extension of a ring R with the same unit $1_R = 1_S$ and every module is unitary. We get the following lemmas easily.

LEMMA 5. *Let N be an R-submodule of an S-module M.*

(1) *Let $N^t = \sum_{i=1}^{t} Nx_i$ where $Nx_i = \{nx_i \mid n \in N\}$. Then N^t is R-submodule of M and N^n is an S-submodule of M.*

(2) *Let $Nx_i^{-1} = \{m \in M \mid mx_i \in N\}$ and $b(N) = \cap Nx_i^{-1}$. Then $b(N)$ is an S-submodule of M.*

(3) *$N^t x_{t+1}^{-1} \cap N$ is an R-submodule of N.*

Proof. (1) Clearly $N^n S \subset N^n$.

(2) Since $(mr_1x_1 + \ldots + mr_nx_n)x_i = mx_1a_1 + \cdots + mx_na_n$, $b(N)$ is an S-submodule.

(3) $(nr)x_{t+1} = n(x_1a_1 + \cdots + x_na_n) \in N$ for any r and some a_j in R.

LEMMA 6. *M contains an R-submodule N maximal with respect to $b(N) = 0$. And if* $\operatorname{rank} M_R = m$, *then* $\operatorname{rank}(M/N)_R \leq m$.

Proof. By Zorn's lemma we can find a maximal R-submodule A_1, $\cdots$, A_t be R-submodules of M strictly containing N whose sum is direct module N. Then $b(A) = 0$ for every j. If $t > m$ then for some j,

$\sum_{i \neq j} b(A_i) \cap b(A_j) \neq 0$. Thus $b((\sum A_i) \cap A_j)) \neq 0$. Hence $\sum_{i \neq j} A_i \cap A_j$ strictly contains N. This is contradiction on the maximality of N. Thus we get $\operatorname{rank}(M/N) \leq m$.

THEOREM 7. *Any S-module M has finfite uniform dimension if and only if it has finite uniform dimension when considered as an R-module.*

Proof. Since every S-module is R-module if an S-module M has finite uniform dimension as R-module then it has also finite uniform dimension as S-module. Conversely, we assume that M has infinite direct sum of R-submodules $\{\sum N_i\}$. Then we can show that M has infinite direct sum of S-submodules $\{K_i\}$. At first, if $(\sum N_i^t)x_{t+1}^{-1} \cap N = 0$ and $\{\sum N_{i_j}^t\}$ is direct sum where $N = \sum N_i$, then $\sum N_i^{t+1}$ is direct sum as R-submodules. For if $\sum x_{i_j} = 0$ where $x_{i_j} \in N_{i_j}^{t+1}$ that is $x_{i_j} = (\sum y_{i_j}) + (\sum n_{i_j})x_{t+1} = 0$ where $y_{i_j} \in N_{i_j}^t$ and $n_{i_j} \in N_{i_j}$. That implies $\sum n_{i_j} \in (\sum N_{i_j}^t)x_{t+1}^{-1} \cap N = 0$. Thus we get $n_{i_j} = 0$ for all i_j and $y_{i_j} = 0$ for $\sum N_{i_j}^t$ is direct sum. If $(\sum N_i^t)x_{t+1}^{-1} \cap N = 0$ for all $1 \leq t \leq n-1$, then N^n is an infinite direct sum of S-submodules. Secondly we assume that for any infinite subcollection $J \subset I$, $(\sum N_j^t)x_{t+1} \cap N \neq 0$ for $1 \leq t \leq n-1$. Then for every $p > 0$ there exist $y \in N_{p+1} + \cdots + N_k$ such that yx_{t+1} is contained in $N_{p+1}^t + \cdots + N_m^t$. Let $q = \max(k, m)$. Then

$$0 \neq y \in (N_{p+1} + \cdots + N_q)^{t+1} \cap (N_{p+1}^t + \cdots + N_q^t)x_{t+1} = W.$$

Thus $W^{t+1} \subset N_{p+1}^t + \cdots + N_q^t$ and $\sum W_i^{t+1}$ is direct. So we get a subcollection $\{\sum K_i\}$ of S-submodules which is infinte direct sum by doing same process continuousely for $1 \leq t \leq n-1$.

From this theorem we get the following corollary.

COROLLARY. *If S is a left Goldie ring, then R is also left Goldie ring.*

The following example shows that some properties do not hold in finite subnormalizing extension.

EXAMPLE 8. Let K be a field and $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ $S = \begin{pmatrix} K & K \\ K & K \end{pmatrix}$.

(1) Let $I = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$. Then $IS = S$. But if S is a finite normalizing extension of a ring R, then $IS \neq S$ for any proper ideal I of R.

(2) Cleary $J(R) = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$ and $J(S) = 0$. Thus $J(R) \neq R \cap J(S)$ which is different from the case of finite normalizing extensions.

References

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