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NOTE ON JOINTLY $*$–PARANORMALITY

YOUNGSIK PARK AND CHEONSEOUNG RYOO

1. Introduction

Throughout this paper, H will always denote a Hibert space. $B(H)$ will denote the algebra of bounded linear operater on H and $B(H^n)$, the set of a commuting n-tuple of operators in $B(H)$, where H^n denotes the orthogonal direct sum of H with itself n times. Let $\sigma(T)$, $\sigma_a(T)$ be the joint spectrum, the joint approximate point spectrum of $T = (T_1, T_2, \cdots, T_n)$ in $B(H)$. We shall define some classes of operator families of $B(H^n)$. $T = (T_1, T_2, \cdots, T_n)$ $(\in B(H^n))$ is called jointly $*$-paranormal if $\|T_i^* x\|^2 \leq \|T_i^2 x\|$ for a unit vetor x in H, $i = 1, 2, \cdots, n$. Similarly T is called jointly hyponomal if $T_i^* T_i \geq T_i T_i^*$ for $i = 1, 2, \cdots, n$. These notions have been considered by A. Athavale ([1]), R.E. Curto, P.S. Muhly and J. Xia ([4]), M. Chō and M. Takaguchi ([3]), M. Chō and A.T. Dash ([2]), and A. Lubin ([6]). From the definition of jointly $*$-paranormality, we have the following:

Let $T = (T_1, T_2, \cdots, T_n)$ be a commuting n-tuple of operators in $B(H)$.

(1) For $n = 1$, definition of jointly $*$-pranormality is the usual definition of a $*$-paranormal on H.

(2) If T is jointly $*$-paranormal, then any subtuple of T is jointly $*$-paranormal.

(3) If T is jointly $*$-paranormal and N is a normal operator commuting with each T_i, then $(NT_1, NT_2, \cdots, NT_n)$ is jointly $*$-paranormal.

(4) If T is jointly $*$-paranormal, then $(I, T_1, T_2^2, \cdots, T_{n-1}^{n-1})$ is jointly $*$-paranormal.

(5) If a single operator T is a $*$-paranormal operator, then $(T, T, \cdots, T)$ is also jointly $*$-paranormal.

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2. Generalization and unitarily equivalence

An operator $T \in B(H)$ is said to be isometric if $\|Tx\| = \|x\|$ for all $x \in H$. It is easy to verify that every isometric operator is a hyponormal operator. An operator $T \in B(H)$ is said to be unitarily equivalent to an operator S if $S = U^*TU$ for a unitary operator U. In [5], T. Furuta and R. Nakamoto have proved the following theorem.

THEOREM A. *A hyponormal operator unitarily equivalent to its adjoint is a normal operator.*

We have a generalized result for a commuting n-tuple of operators as following lemma.

LEMMA 2.1. *Let $T = (T_1, T_2, \cdots, T_n)$ be a commuting n-tuple of operators such that each T_i is unitarily equivalent to its adjoint. If T is jointly hyponormal, then T is jointly normal.*

Proof. Suppose T_i is hyponormal and $T_i = U^*T_i^*U$ for a unitary operator U $(i = 1, 2, \cdots, n)$. Since T is jointly hyponormal, it follows that

$$\begin{aligned} \|T_i^*x\| \leq \|T_ix\| &= \|U^*T_i^*U\| = \|T_i^*Ux\| \\ &\leq \|T_iUx\| = \|U^*T_iUx\| = \|T_i^*x\| \end{aligned}$$

for any x in H ($i = 1, 2, \cdots, n$). Thus each T_i is a normal operator.

We generalize the above theorem and prove similar results for classes of jointly hyponormal and jointly $*$-paranormal. The following theorem is proved by Lemma 2.1.

THEOREM 2.2. *Let $T = (T_1, T_2, \cdots, T_n)$ be a commuting n-tuple of operators such that each T_i is unitarily equivalent to a hyponormal operator. Then T is jointly hyponormal.*

THEOREM 2.3. *Let $T = (T_1, T_2, \cdots, T_n)$ be a commuting n-tuple of operators such that T_i is unitarily equivelent to a $*$-paranormal operator. Then T is jointly $*$-paranormal.*

Proof. It is sufficient to show that each T_i is a $*$-paranormal operator. Suppose that $T_i = U^*SU$ for a $*$-paranormal operator S and a unitary operator U, $i = 1, 2, \cdots, n$. Now for each $x \in H$, we have

$$\begin{aligned}\|T_i^* x\|^2 &= \|U^* S^* U x\|^2 = \|S^* U x\|^2 \\ &\le \|S^2 U x\| \, \|U x\| = \|U T_i^2 x\| \, \|x\| \\ &\le \|T_i^2 x\| \, \|x\|,\end{aligned}$$

$i = 1, 2, \cdots, n$. Therefore each T_i is a $*$-paranormal operator.

The following corollaries follow from Theorem 2.3.

COROLLARY 2.4. *Let $T = (T_1, T_2, \cdots, T_n)$ be a commmuting n-tuple of operators and let $S = (S_1, S_2, \cdots, S_n)$ be jointly $*$-paranormal. If each T_i is unitarily equivalent to S_i for $i = 1, 2, \cdots, n$,then T is jointly $*$-paranormal.*

COROLLARY 2.5. *Let $T = (T_1, T_2, \cdots, T_n)$ be jointly $*$-paranormal. Then $(U^*T_1U, U^*T_2U, \cdots, U^*T_n)$ is jointly $*$-paranormal, where U is any unitary operator.*

The product of two commuting $*$-paranormal operators, in general, may not be a $*$-paranormal operator. In [7], S.M. Patel has proved the following theorem.

THEOREM B. *Let A be a hyponormal operator and let B be a $*$-paranormal operator. If A and B are doubly commutative, then AB is a $*$-paranormal operator.*

We charaterize the above theorem and prove a similar result for the class of $*$-paranormal operators.

THEOREM 2.6. *Let $T = (T_1, T_2, \cdots, T_n)$ be a commuting n-tuple of operators such that each T_i commutes with an isometric operator S. If T is jointly $*$-paranormal, then $(T_1S, T_2S, \cdots, T_nS)$ is jointly $*$-paranormal.*

Proof. Let x be a unit vetor in H. If T is jointly $*$-paranormal and S is an isometric operator, then we have

$$\begin{aligned}\|(T_iS)^*x\|^2 &= \|S^*T_i^*x\|^2 \le \|ST_i^*x\|^2 = \|T_i^*x\|^2 \\ &\le \|T_i^2x\| = \|ST_i^2x\| = \|ST_i^2xS\| = \|(T_iS)^2x\|.\end{aligned}$$

Thus each T_iS is a $*$-paranormal operator and T_iS commutes with T_jS for $i, j = 1, 2, \cdots, n$. Therefore $(T_1S, T_2S, \cdots, T_nS)$ is jointly $*$-paranormal.

There exists an example that the product of two double commuting n-tuples of $*$-paranormal operators is jointly $*$-paranormal by the following given conditions.

EXAMPLE 2.7. Let $T = (T_1, T_2, \cdots, T_n)$ and $S = (S_1, S_2, \cdots, S_n)$ be jointly $*$-paranormal such that T_i and S_i are doubly commutative for each i.

(1) If $\|T_i^*S_ix\|\,\|x\| \geq \|T_i^*x\|\,\|S_ix\|$ for all $x \in H$ and for each i, $i = 1, 2, \cdots, n$, then $(T_1S_1, T_2S_2, \cdots, T_nS_n)$ is jointly $*$-paranormal.

(2) If $\|T_i^*S_i^2x\|\,\|x\| \geq \|T_i^*x\|\,\|S_i^2x\|$ for all $x \in H$ and for each i, $i = 1, 2, \cdots, n$, then $(T_1S_1, T_2S_2, \cdots, T_nS_n)$ is jointly $*$-paranormal.

Proof. (1) : Assume that $\|T_i^*S_ix\|\,\|x\| \geq \|T_i^*x\|\,\|S_ix\|$ for all $x \in H$ and for each i, $i = 1, 2, \cdots, n$. Since T_i and S_i are doubly commuting $*$-paranormal operators, we have

$$\begin{aligned}
&\|T_i^2S_i^2x\|\,\|S_i^2x\|\,\|S_ix\|^2\|T_i^*x\|\,\|x\|^2 \\
&\qquad \geq \|T_i^*S_i^2x\|\,\|S_ix\|^2\|T_i^*x\|\,\|x\|^2 \\
&\qquad \geq \|S_i^*T_i^*x\|^2\|T_i^*S_i^2x\|\,\|S_ix\|^2\|x\|^2 \\
&\qquad \geq \|S_i^*T_i^*x\|^2\|T_i^*S_ix\|\,\|S_i^2x\|\,\|S_ix\|\,\|x\|^2 \\
&\qquad \geq \|S_i^*T_i^*x\|^2\|T_i^*x\|\,\|S_ix\|^2\|S_i^2x\|\,\|x\|.
\end{aligned}$$

Thus each T_iS_i is a $*$-paranormal operator and $(T_1S_1, T_2S_2, \cdots, T_nS_n)$ is a commuting n-tuple of operators. Therefore $(T_1S_1, T_2S_2, \cdots, T_nS_n)$ is jointly $*$-paranormal.

(2) : By similar method we have

$$\begin{aligned}
\|T_i^2S_i^2x\|\,\|S_i^2x\|\,\|S_i^*x\|\,\|T_i^*x\|\,\|x\| &\geq \|T_i^*S_i^2x\|^2\|S_i^*x\|\,\|T_i^*x\|\,\|x\| \\
&\geq \|S_i^*T_i^*x\|^2\|T_i^*S_i^2x\|\,\|S_i^*x\|\,\|x\| \\
&\geq \|S_i^*T_i^*x\|^2\|T_i^*x\|\,\|S_i^2x\|\,\|S_i^*x\|.
\end{aligned}$$

Thus each T_iS_i is a $*$-paranormal and so $(T_1S_1, T_2S_2, \cdots, T_nS_n)$ is jointly $*$-paranormal.

In [8], C.R. Putnam has proved the following theorem.

Theorem C. *Let T be a hyponormal operator and let z belong to the boundary of $\sigma(T)$. Then $|z| \in \sigma(T^*T)^{\frac{1}{2}} \bigcap \sigma(TT^*)^{\frac{1}{2}}$.*

LEMMA 2.8. *Let $T = (T_1, T_2, \cdots, T_n)$ be a commuting n-tuple of operators. If T is jointly $*$-paranormal and λ_i $(\neq 0) \in \sigma_a(T_i)$ for each i, $i = 1, 2, \cdots, n$, then $\overline{\lambda_i} \in \sigma_a(T_i^*)$.*

Proof. Let λ_i $(\neq 0) \in \sigma_a(T_i)$ for each i, $i = 1, 2, \cdots, n$. Then there is a sequence $\{x_n\}$ of unit vectors such that $\|(T_i - \lambda_i)x_n\| \to 0$ as $n \to \infty$. Since each T_i is a $*$-paranormal operator, we have

$$\|(T_i^* - \overline{\lambda_i})x_n\|^2 \leq \|T_i^2 x_n\| - \lambda_i(x_n, T_i x_n) - \overline{\lambda_i}(T_i x_n, x_n) + |\lambda_i|^2 .$$

The right term of above inequality goes to zero as n goes to infinty. Thus $\overline{\lambda_i} \in \sigma_a(T_i^*)$.

We generalize the above theorem and have a similar result in the following theorem.

THEOREM 2.9. *Let $T = (T_1, T_2, \cdots, T_n)$ be a commuting n-tuple of operators and let $z = (z_1, z_2, \cdots, z_n) \in \sigma_a(T)$. If T is jointly $*$-paranormal. then $|z_i| \in \sigma(T_i^*T_i)^{\frac{1}{2}} \bigcap \sigma(T_iT_i^*)^{\frac{1}{2}}$, $i = 1, 2, \cdots, n$.*

Proof. Let $z = (z_1, z_2, \cdots, z_n) \in \sigma_a(T)$. Then there is a sequence $\{x_n\}$ of unit vectors such that

$$\|(T_i - z_i)x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Since T is jointly $*$-paranormal, by Lemma 2.8 we have

$$\|(T_i^* - \overline{z_i}x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus it follows that

$$\|(T_i^*T_i - |z_i|^2 I)x_n\| \to 0$$

and

$$\|(T_iT_i^* - |z_i|^2 I)x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

So

$$\|((T_i^*T_i)^{\frac{1}{2}} - |z_i| I)x_n\| \to 0,$$

$$\|((T_iT_i^*)^{\frac{1}{2}} - |z_i| I)x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore $|z_i| \in \sigma(T_i^*T_i)^{\frac{1}{2}} \bigcap \sigma(T_iT_i^*)^{\frac{1}{2}}$ for each i.

In the following example, we know that if we replace $\sigma_a(T)$ by $\sigma(T)$, then Theorem 2.9 deoes not hold.

EXAMPLE 2.10. Let $T = (T_1, T_2, \cdots, T_n)$ be an n-tuple of operators such that $T_i = iT_1$ for each i, $i = 1, 2, \cdots, n$ and T_1 is the unilateral shift operator defined by

$$T_1(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$$

for a sequence $\{x_n\}$ in l^2. Then we have

$$T_i^* T_i = i^2 I, \qquad T_i T_i^* = \text{diag}(0, i^2, i^2, \cdots)$$

for each i, $i = 1, 2, \cdots, n$. Since $\|T_i^* x\|^2 \leq \|T_i^2 x\|$ for any unit vector x and for each i, $i = 1, 2, \cdots, n$, each T_i is a $*$-paranormal operator. And so T is jointly $*$-paranormal. Also we can see that $\sigma(T_i) = \{\lambda \in C : |\lambda| \leq i\}$. By simple calculations we have $\sigma(T_i^* T_i)^{\frac{1}{2}} = \{i\}$ and $\sigma(T_i T_i^*)^{\frac{1}{2}} = \{0, i\}$ for each i. Therefore the condition $z = (z_1, z_2, \cdots, z_n) \in \sigma(T)$ does not imply $|z_i| \in \sigma(T_i^* T_i)^{\frac{1}{2}} \bigcap \sigma(T_i T_i^*)^{\frac{1}{2}}$, $i = 1,2,\cdots,n$.

References

1. A. Athavala, *On joint hyponormality of operators*, Proc. Amer. Math. Soc **103** (1988), 417–423
2. M. Chō and A.T. Dash, *On the joint spectra of doubly commuting n tuples of semi-normal operators*, Glasgow Math. J. **26** (1985), 47–50.
3. M. Chō and M. Takaguchi, *Some classes of commuting n-tuples of operators*, Studia Mathemati **LXXX** (1984), 245–259.
4. R.E. Curto, P.S Muhly and J. Xia, *Hyponormal pairs of commuting operators*, to appear Op Th. Adn.and Appl.
5. T. Furta and R Nakamoto, *On some theorems of Berberian and Sheth*, Proc. Japan Academy **46** (1970), 841–845
6. A. Lubin, *Weighted shifts and commuting normal extension*, J Austral. Math Soc. **27** (1979), 17–26.
7. S.M Patel, *Contributions to the study of sprctraloid operators*, Ph D. Thesis, Del-phi University, 1974.
8. C.R. Putnam, *Spectra of polar factors of hyponormal operators*, Trans. Amer. Math. Soc. **188** (1974), 419–428

Department of Mathematics
University of Ulsan
Ulsan 680–749, Korea

Department of Mathematics
Kyungpook National University
Taegu 702–701, Korea