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CONTROLLABILITY FOR RETARDED SYSTEM WITH NONLINEAR TERM IN HILBERT SPACE

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1. Introduction

In this paper we deal with control problem for semilinear parabolic type equation in Hilbert space H as follows.

$$\begin{aligned}\frac{d}{dt}x(t) =& A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds \\ &+ f(t, x(t)) + B_0u(t), \quad t \in (0, T].\end{aligned} \tag{1.1}$$

Let A_0 be the operator associated with a sesquilinear form defined on $V \times V$ satisfying Gårding's inequality and the operators A_1 and A_2 be a bounded linear operators from V to V^*. With the aid of the solution semigroup and fundamental solution of (1.1) that was constructed in [9] the equation (1.1) can be transformed onto an abstract equation

$$\frac{d}{dt}z(t) = Az(t) + F(z(t)) + Bu(t) \tag{1.2}$$

in the product space $Z = H \times L^2(-h, 0; V)$. The results of this paper is made up two parts: The first is to give wellposedness and regularity in section 2. We will give the result by using the intermediate property and contraction mapping principle. Next, by assuming that the system of generalized eigenspaces of A is complete in Z, we establish necessary and sufficient condition for the approximate controllability for (1.2) in section 3. Our purpose is to seek the condition for linear system where $F \equiv 0$ and the equivalence of the controllability for between the semilinear system and the associated linear system.

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2. Wellposedness and regularity

We consider the problem of control for the following retarded functional differential equation of parabolic type with nonlinear term

$$\begin{aligned} (2.1) \qquad \frac{d}{dt}x(t) = A_0x(t) + A_1x(t-h)\int_{-h}^{0} a(s)A_2x(t+s)ds \\ + f(t,x(t)) + B_0u(t), \end{aligned}$$

$$(2.2) \qquad x(0) = g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0).$$

in Hilbert space in H. Let V be another Hilbert space such that $V \subset H \subset V^*$. The notations $|\cdot|$, $||\cdot||$ denote the norms of H, V respectively as usual. Let A_0 be the operator associated with a sesquilinear form

$$(A_0u, v) = -a(u, v), \quad u,\ v \in V,$$

where $a(u, v)$ is a bounded sesquilinear form in $V \times V$ satisfying Gårding's inequality. The operators A_1 and A_2 are bounded linear operators from V to V^* such that they map $D(A_0)$ into H. We may assume that $(D(A_0), H)_{1/2,2} = V$ satisfying

$$(2.3) \qquad ||u|| \leq C_1||u||_{D(A_0)}^{1/2}|u|^{1/2}$$

for some a constant $C_1 > 0$ where $(D(A_0), H)_{\theta,p}$ denotes the real interpolation space between $D(A_0)$ and H (see R. Seeley ([6])). The function $a(\cdot)$ is assume to be a real valued Hölder continous in $[-h, 0]$. B_0 is a bounded linear operator from some Banach space U to H. Replacing intermidiate space F in the paper [1] with the space H, we can derive the results of G. Blasio, K. Kunisch and E. Sinestrari ([1]) regading term by term to deduce the following result.

PROPOSITION 2.1. *Let $g = (g^0, g^1) \in Z = H \times L^2(-h, 0; V)$ and $f \in L^2(0, T; V^*)$. Then for each $T > 0$, a solution x of the equation (2.1) and (2.2) belongs to*

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

Moreover, for some constant C_T we have

$$\begin{aligned} ||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq & C_T(|g^0| + ||g^1||_{L^2(-h,0;V)} \\ & + ||f||_{L^2(0,T;V^*)} + ||u||_{L^2(0,T;U)}). \end{aligned}$$

THEOREM 2.1. *Let f be a nonlinear mapping from $\mathcal{R} \times V$ into H and assume that for any x_1, $x_2 \in V$ there exists a constant $L > 0$ such that*

$$|f(t,x_1) - f(f,x_2)| \le L||x_1 - x_2||$$
$$f(t,0) = 0.$$

Then there exists a unique solution x of (2.1) and (2.2) such that

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

for any $g = (g^0, g^1) \in Z = H \times L^2(-h;V)$. Moreover, there exists a constant C such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C(|g^0| + ||g^1||_{L^2(-h,0,V)} + ||u||_{L^2(0,T;U)}).$$

Proof. Let us fix $T \in (0,h)$ such that

$$C_1 C_T L(T/\sqrt{2})^{\frac{1}{2}} < 1. \tag{2.4}$$

For $i = 1$ 2, we consider the following equation.

$$\begin{aligned} \frac{d}{dt}y_i(t) &= A_0 y_i(t) + A_1 y_i(t-h)\int_{-h}^{0} a(s)A_2 y_i(t+s)ds \\ &\quad + f(t, x_i(t)) + B_0 u(t), \qquad t \in (0,T] \\ y_i(0) &= g^0, \quad y_i(s) = g^1(s), \qquad s \in [-h, 0). \end{aligned}$$

From theorem 3.3 of [1] and uniformly Lipschitz continuous of f it follows that

$$\begin{aligned} ||y_1 - y_2||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} &\le C_T ||f(\cdot,x_1) - f(\cdot,x_2)||_{L^2(0,T;H)}, \\ ||f(\cdot,x_1) - f(\cdot,x_2)||_{L^2(0,T;H)} &\le L||x_1 - x_2||_{L^2(0,T;V)}. \end{aligned}$$

Using the Hölder inequality we also obtain that

$$||y_1 - y_2||_{L^2(0,T;H)} \le \frac{\sqrt{T}}{2}||y_1 - y_2||_{W^{1,2}(0,T;H)}, \tag{2.5}$$

and hence, from (2.3) and (2.5)

$$||y_1 - y_2||_{L^2(0,T;V)} \le C_1 C_T L(T/\sqrt{2})^{\frac{1}{2}} ||x_1 - x_2||_{L^2(0,T;V)}.$$

So by virtue of the condition (2.4) the contraction principle gives that the equation of (2.1) and (2.2) has a unique solution in $[-h, T]$.

Let $x(\cdot)$ be a solution of (2.1) and (2.2) and $y(\cdot)$ be a solution of following equation.

$$\frac{d}{dt} y(t) = A_0 y(t) + A_1 y(t-h) + \int_{-h}^{0} a(s) A_2 y(t+s) ds + B_0 u(t), \qquad t \in (0, T]$$

$$y(0) = g^0, \quad y(s) = g^1(s), \qquad s \in [-h, 0).$$

In virtue of Theorem 3.3 ([1]) we have

$$\begin{aligned} ||x - y||_{L^2(0,T;D(A_0)) \cap W^{1,2}(0,T,H)} &\le C_T ||f(\cdot, x)||_{L^2(0,T;H)} \\ &\le C_T L(||x - y||_{L^2(0,T;V)} + ||y||_{L^2(0,T,V)}). \end{aligned}$$

Combining (2.3), (2.5) and above inequality we have

$$\begin{aligned} ||x - y||_{L^2(0,T;V)} &\le C_1 ||x - y||^{\frac{1}{2}}_{L^2(0,T;D(A_0))} ||x - y||^{\frac{1}{2}}_{L^2(0,T;H)} \\ &\le C_1 (T/\sqrt{2})^{\frac{1}{2}} C_T L(||x - y||_{L^2(0,T;V)} + ||y||_{L^2(0,T;V)}). \end{aligned}$$

Therefore, we have

$$(2.6) \qquad \begin{aligned} ||x - y||_{L^2(0,T;V)} &\le \frac{C_1 C_T L (T/\sqrt{2})^{\frac{1}{2}}}{1 - C_1 C_T L (T/\sqrt{2})^{\frac{1}{2}}} ||y||_{L^2(0,T,V)}, \\ ||x||_{L^2(0,T;V)} &\le \frac{1}{1 - C_1 C_T L (T/\sqrt{2})^{\frac{1}{2}}} ||y||_{L^2(0,T,V)}. \end{aligned}$$

Combining Proposition 2.1 and (2.6) we obtain

$$\begin{aligned} ||x||_{L^2(0,T,V) \cap W^{1,2}(0,T;V^*)} &\le C_T(|g_0| + ||g^1||_{L^2(0,T,V)} + L||x||_{L^2(0,T;V)} \\ &\quad + ||u||_{L^2(0,T:U)}) \\ &\le C_T(|g_0| + ||g^1||_{L^2(0,T;V)} + ||u||_{L^2(0,T:U)} \\ &\quad + \frac{L}{1 - C_1 C_T L (T/\sqrt{2})^{\frac{1}{2}}} ||y||_{L^2(0,T;V)}) \\ &\le C(|g_0| + ||g^1||_{L^2(0,T;V)} + ||u||_{L^2(0,T:U)}). \end{aligned}$$

Since the condition (2.4) is independent of initial value, the solution of (2.1) and (2.2) can be extended to the interval $[-h, nT]$ for n is a natual number, and so the proof is complete.

3. Approximate controllability

In this section we consider the approximate controllability of retardsed system with nonmlinear term. Our method is derived to the relation between nonlinear system and linear system. Let $Z = H \times L^2(-h, 0; V)$ be the state space and be a product Hilbert space with the norm

$$||g||_Z = (|g^0|^2 + \int_{-h}^{0} ||g^1(s)||^2 ds)^{\frac{1}{2}}, \quad g = (g^0, g^1) \in Z.$$

Let $g \in Z$ and $x(t; g, f, B_0 u)$ be a solution of the equation (2.1) and (2.2) associated with nonlinear term f and control $B_0 u$ at time t. In view of the result of Theorem 2.1 considered as an equation in V^*, we can define the solution semigroup for the problem (2.1) and (2.2) as follows:

$$S(t)g = (x(t; g, 0, 0), x_t(\cdot; g, 0, 0)) \tag{3.1}$$

where $g = (g^0, g^1) \in Z$, $x(t; g, 0, 0)$ is the solution of (2.1) and (2.2) with $f(t, x) = 0$ and $B_0 = 0$ and $x_t(s; g, 0, 0) = x(t + s; g, 0, 0)$ defined in $[-h, 0]$.

For the sake of simplicity, we assume that $S(t)$ is uniformly bounded, that is, there exists a constant $M \geq 1$ such that

$$||S(t)||_Z \leq M. \tag{3.2}$$

The equation (2.1) and (2.2) can be transformed into an abstract equation:

$$z(t) = Az(t) + F(z(t)) + Bu(t), \tag{3.3}$$

$$z(0) = g, \tag{3.4}$$

where $z(t) = (x(t), x_t(\cdot))$ belongs to the Hilbert space Z and $g = (g^0, g^1) \in Z$. The operator A is the infinitesimal generator of C_0-semigroup $S(t)$, $F(z(t)) = (f(t, x(t)), 0)$ and $Bu = (B_0 u, 0)$. The mild solution of initial problem (3.3) and (3.4) is the following form:

$$z(t; g, f, Bu) = S(t)g + \int_0^t S(t-s)F(z(s))ds + \int_0^t S(t-s)Bu(s)ds.$$

Let $\sigma_p(A)$ be the point spectrum of A and $\sigma_0(A)$ be the set of all poles of $(\mu - A)^{-1}$. Let λ be a pole of the resolvent of A of order k_λ. Then the generalized eigenspace corresponding to λ is given by

$$Z_\lambda \equiv \mathrm{Ker}(\lambda I - A)^{k_\lambda}.$$

We define reachable sets for the semilinear system of (3.3) and (3.4), and the linear system in case where $f \equiv 0$. For $T > 0$, $g \in Z$ and $u \in L^2(0, T; U)$ we set

$$\begin{aligned} L_T(g) &= \{z(T; g, 0, Bu) : u \in L^2(0, T; U)\}, \\ R_T(g) &= \{z(T; g, f, Bu) : u \in L^2(0, T; U)\}, \\ L(g) &= \cup_{T>0} L_T(g), \quad R(g) = \cup_{T>0} R_T(g). \end{aligned}$$

Since Z is reflexive space, we can consider the dual system of the problem (3.3) and (3.4). Moreover we introduce the unobservable subspace for the dual system of the problem (3.3) and (3.4):

$$N_T = \cap_{0 \le t \le T} B^* S^*(t), \qquad N = \cap_{T>0} N_T.$$

Here, the adjoint operator B^* of B is defined by

$$B^*\phi = B_0^*\phi^0, \quad \phi = (\phi^0, \phi^1) \in Z^*.$$

It is known that $L_T(0)$ is independent of T in case where the controller B is identity operator.

DEFINITION 3.1. (the case linear system, that is, $F \equiv 0$) For any $\lambda \in \sigma_p(A)$,

(1) the system of (3.3) and (3.4) is said to be Z_λ-controllable (resp. in time T) for initial value g if $Z_\lambda \subset \overline{L(g)}$ (resp. $Z_\lambda \subset \overline{L_T(g)}$).
(2) The dual system of (3.3) and (3.4) is said to be $Z^*_{\bar{\lambda}}$-oberservable (resp. in the time T) if $N \cap Z^*_{\bar{\lambda}} = \{0\}$ (resp. $N_T \cap Z^*_{\bar{\lambda}} = \{0\}$),

where $Z^*_{\overline{\lambda}}$ is the generalized eigenspace for $\overline{\lambda}$ which is an eigenvalue of A^* and the symbol bar denotes the closure.

We can also define approximate controllability for the semilinear system in case where $F \neq 0$ by replacing the reachable set L_T by R_T. From using the duality theorem we obtain that the system of (3.3) and (3.4) is Z_λ-controllable in time T in Z if and only if the system of (3.3) and (3.4) is Z_λ-controllable in Z, that is, independent of the time.

DEFINITION 3.2. The system of generalized eigenspaces of A is complete in Z if

$$\mathrm{Cl}(\cup_{\lambda \in \sigma_p(A)} Z_\lambda) = Z.$$

DEFINITION 3.3. (the linear case)

(1) The system of (3.3) and (3.4) is said to be Z-approximately controllable for initial value g (resp. in time T) if $\overline{L(g)} = Z$ (resp. $\overline{L_T(g)} = Z$).

(2) The dual system of (3.3) and (3.4) is Z^*-controllable (resp. in time T) if $N = \{0\}$ (resp. $N_T = \{0\}$).

Let us assume that $\sigma_p(A) = \sigma_0(A)$ and the system of generalized eigenspaces of A is complete in Z. Then it is easily seen that

$$\overline{L_T(g)} = Z \Longleftrightarrow \overline{L(g)} = Z \Longleftrightarrow \overline{L_T(0)} = Z \tag{3.5}$$

for any $g \in Z$. In what follows we assume that f is uniformly bounded on $[0, \infty) \times H$, that is, there exists a positive constant M such that

$$|f(t,x)| \leq M \tag{3.6}$$

for each $t \geq 0$ and $x \in H$, If the assumption (3.6) is satisfied then $||f(z)||_Z \leq M$ for any $z \in Z$. We can prove following Theorem by modifying an argument in [10].

THEOREM 3.1. *Let us assume that the condition (3.6), $\sigma_p(A) = \sigma_0(A)$ and the system of generalized eigenspaces of A is complete in Z. Then for any $g \in Z$, $\overline{L_T(g)} = Z$ if and only if $\overline{R_T(g)} = Z$.*

Proof. (Sufficiency) Since $\overline{L_T(0)}$ is a closed linear subspace there exists a $z_0 \in Z$ such that $||z_0|| = 1$ and $\inf\{||z_0 - z|| : z \in L_T(0)\} > \frac{1}{2}$. Let $g \in Z$ be arbitrary and let

$$|r| > 2(M||g||_Z + M^2T). \tag{3.7}$$

Then $rz_0 \notin \overline{R_T(g)}$. In fact, combining (3.6) and (3.7) we obtain that

$$\begin{aligned}||z(t;g,f,Bu)-rz_0||_Z &\geq ||\int_0^t S(t-s)Bu(s)ds - rz_0||_Z - ||S(t)g||_Z \\ &\quad - ||\int_0^t S(t-s)F(z(s))ds||_Z \\ &\geq \frac{r}{2} - M||g||_Z - M^2T \\ &> 0.\end{aligned}$$

(Necessity) Let $\epsilon > 0$ and $z_0 \in Z$ and let $\delta \leq \frac{1}{2}\epsilon M^2$. Put $z_0(s) = z(s;g,f,0)$ and $z_1 = z(T-\delta;g,f,0)$, where $z(T-\delta;g,f,0) = S(T-\delta)g + \int_0^{T-\delta} S(T-\delta-s)F(z_0(s))ds$. Consider the following problem:

$$\begin{aligned}&\frac{d}{dt}y(t) = Ay(t) + Bu(t), \quad T-\delta < s \leq T, \\ &y(T-\delta) = z_1.\end{aligned}$$

Then from (3.5) there exists $u_1 \in L^2(T-\delta, T; U)$ such that

$$||y_{u_1}(T) - z_0|| < \frac{\epsilon}{2} \tag{3.8}$$

where $y_{u_1}(T) = S(\delta)z_1 + \int_{T-\delta}^T S(T-s)Bu_1(s)ds$. Now we set

$$v(s) = \begin{cases} 0, & \text{if } s \leq T-\delta, \\ u_1(s), & \text{if } T-\delta < s \leq T. \end{cases}$$

Then $v \in L^2(0,T;U)$ and from (3.6) and (3.8) we obtain that

$$\begin{aligned}||z(T;g,f,Bv) - z_0||_Z &\leq ||S(\delta)z_1 + \int_{T-\delta}^T S(T-s)Bu_1(s)ds - z_0||_Z \\ &\quad + ||\int_{T-\delta}^T S(T-s)F(z_{u_1}(s))ds||_Z \\ &< \frac{\epsilon}{2} + M^2\delta \leq \epsilon.\end{aligned}$$

Hence the proof is complete.

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