Pusan Kyŏngnam Math. J. 8(1992), No. 2, pp. 163–170

FUZZY Γ–RINGS

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The concept of a fuzzy set, introduced by Zadeh ([6]), was applied in [2] to generalize some of the basic concepts of general topology. Rosenfeld ([5]) applied this concept to the theory of groupoids and groups. The present paper constitutes a similar application to the elementary theory of Γ-rings.

We recall that a fuzzy set in a set S is a function μ from S into [0, 1]. Let μ and ν be fuzzy sets in a set S. Then we define

$$\mu = \nu \Longleftrightarrow \mu(x) = \nu(x) \qquad \text{for all } x \in S.$$

$$\mu \subseteq \nu \Longleftrightarrow \mu(x) \leq \nu(x) \qquad \text{for all } x \in S.$$

$$(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\} \qquad \text{for all } x \in S.$$

$$(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} \qquad \text{for all } x \in S.$$

More generally, for a family of fuzzy sets, $\{\mu_i | i \in I\}$, we define

$$(\cup\mu_i)(x) = \sup_{i\in I}\{\mu_i(x)\}, \quad x \in S$$

$$(\cap\mu_i)(x) = \inf_{i\in I}\{\mu_i(x)\}, \quad x \in S.$$

DEFINITION 1. ([1]) If $M = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ are additive abelian groups, and for all x, y, z in M and all α, β in Γ, the following conditions are satisfied

(1) $x\alpha y$ is an element of M,
(2) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$,
(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

then M is called a Γ-ring.

Received October 7, 1992.

DEFINITION 2. ([1]) A subset A of the Γ-ring M is a left (right) ideal of M if A is an additive subgroup of M and

$$M\Gamma A = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in A\}(A\Gamma M)$$

is contained in A. If A is both a left and a right ideal, then A is a two-sided ideal, or simply an ideal of M.

DEFINITION 3. A fuzzy set μ in a Γ-ring M is called a fuzzy left (right) ideal of M if

(4) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,

(5) $\mu(x\alpha y) \geq \mu(y) \quad (\mu(x\alpha y) \geq \mu(x))$,

for all $x, y \in M$ and all $\alpha \in \Gamma$.

A fuzzy set μ in a Γ-ring M is called a fuzzy ideal of M if μ is both a fuzzy left and a fuzzy right ideal of M.

We note that μ is a fuzzy ideal of M if and only if

(4) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,

(6) $\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$,

for all $x, y \in M$ and all $\alpha \in \Gamma$.

Throughout this paper, all proofs are going to proceed the only left cases, because the right cases are obtained from similar method. We denote 0_M the zero element of a Γ-ring M.

PROPOSITION 1. *If μ is a fuzzy left (right) ideal of a Γ-ring M, then*

(7) $\mu(0_M) \geq \mu(x)$,

(8) $\mu(-x) = \mu(x)$,

(9) $\mu(x-y) = \mu(0_M)$ *implies* $\mu(x) = \mu(y)$,

for all $x, y \in M$.

Proof. (7) We have that for any $x \in M$,

$$\mu(0_M) = \mu(x-x) \geq \min\{\mu(x), \mu(x)\} = \mu(x).$$

(8) By (7), we have that

$$\mu(-x) = \mu(0_M - x) \geq \min\{\mu(0_M), \mu(x)\} = \mu(x)$$

for all $x \in M$. Since x is arbitrary, we conclude that $\mu(-x) = \mu(x)$.

(9) Assume that $\mu(x-y)=\mu(0_M)$ for all $x,y\in M$. Then

$$\begin{aligned}\mu(x)&=\mu(x-y+y)\\&\geq\min\{\mu(x-y),\mu(y)\}\\&=\min\{\mu(0_M),\mu(y)\}\\&=\mu(y).\end{aligned}$$

Similarly, using $\mu(y-x)=\mu(x-y)=0$, we have $\mu(y)\geq\mu(x)$.

EXAMPLE 1. If G and H are additive abelian groups and $M=Hom(G,H),\Gamma=Hom(H,G)$ then M is a Γ-ring with the operations pointwise addition and composition of homomorphisms ([1]). Define a fuzzy set $\mu:M\to[0,1]$ by $\mu(0_M)=t_1,\mu(f)=t_2,0\leq t_2<t_1\leq 1$, where f is any member of M with $f\neq 0_M$. Routine calculations give that μ is a fuzzy left (right) ideal of M.

THEOREM 1. *If μ is a fuzzy left (right) ideal of* a *Γ-ring M, then the set*

$$A:=\{x\in M|\mu(x)=\mu(0_M)\}$$

is a left (right) ideal of M.

Proof. Let $x,y\in A$. Then by (4),

$$\mu(x-y)\geq\min\{\mu(x),\mu(y)\}=\mu(0_M).$$

It follows from (7) that $\mu(x-y)=\mu(0_M)$, so that $x-y\in A$. This means that A is an additive subgroup of M. Now let $u\in A$, $\alpha\in\Gamma$ and $x\in M$. Then by (5), $\mu(x\alpha u)\geq\mu(u)=\mu(0_M)$ and so $\mu(x\alpha u)=\mu(0_M)$. Therefore $x\alpha u\in A$. This completes the proof.

THEOREM 2. *The intersection of any family of fuzzy left (right) ideals of* a *Γ-ring M is also a fuzzy left (right) ideal of M.*

Proof. Let $\{\mu_i\}$ be a family of fuzzy left ideals of a Γ-ring M. Then for every $x,y\in M$ and $\alpha\in\Gamma$,

$$\begin{aligned}(\cap\mu_i)(x-y)&=\inf\{\mu_i(x-y)\}\\&\geq\inf\{\min\{\mu_i(x),\mu_i(y)\}\}\\&=\min\{\inf\mu_i(x),\inf\mu_i(y)\}\\&=\min\{(\cap\mu_i)(x),(\cap\mu_i)(y)\}\end{aligned}$$

and

$$\begin{aligned}(\cap\mu_i)(x\alpha y) &= \inf\{\mu_i(x\alpha y)\}\\ &\geq \inf\{\mu_i(y)\}\\ &= (\cap\mu_i)(y).\end{aligned}$$

DEFINITION 4. ([3]) Let μ be a fuzzy set in a set S. For $t \in [0,1]$, the set

$$\mu_t := \{x \in S | \mu(x) \geq t\}$$

is called a level subset of μ.

THEOREM 3. *Let μ be a fuzzy set in a Γ-ring M. Then*

(a) *if μ is a fuzzy left (right) ideal of M, then μ_t is a left (right) ideal of M for all $t \in [0, \mu(0_M)]$ which is called the level left (right) ideal of M.*

(b) *if μ_t is a left (right) ideal of M for all $t \in Im(\mu)$, then μ is a fuzzy left (right) ideal of M.*

Proof. (a) Assume that μ is a fuzzy left ideal of M. Let $x, y \in \mu_t$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. It follows that

$$\mu(x-y) \geq \min\{\mu(x), \mu(y)\} \geq t,$$

and that $x - y \in \mu_t$. Now let $x \in M$, $\alpha \in \Gamma$ and $y \in \mu_t$. Since μ is a fuzzy left ideal, $\mu(x\alpha y) \geq \mu(y) \geq t$. Thus $x\alpha y \in \mu_t$. Therefore μ_t is a left ideal of M.

(b) Let μ_t be a left ideal of M. We must prove that (4) and (5) hold. If (4) is not true, then

$$\mu(x-y) < \min\{\mu(x), \mu(y)\}$$

for some $x, y \in M$. For these elements x, y, there exist $t_i, t_j \in Im(\mu)$, say $t_i < t_j$, such that $\mu(x) = t_i, \mu(y) = t_j$. Then

$$\mu(x-y) < \min\{\mu(x), \mu(y)\} = t_i,$$

and so $x - y \notin \mu_{t_i}$. This is a contradiction. If (5) is not true, then for a fixed $\alpha \in \Gamma$, there exist $x, y \in M$ such that $\mu(x\alpha y) < \mu(y)$. Let $s_i, s_j \in Im(\mu)$ be such that $s_i < s_j$, $\mu(x) = s_i$ and $\mu(y) = s_j$. Then $\mu(x\alpha y) < \mu(y) = s_j$ and so $x\alpha y \notin \mu_{s_j}$, a contradiction. This completes the proof.

THEOREM 4. *Let A be a left (right) ideal of a Γ-ring M. Then for any $t \in (0,1)$, there exists a fuzzy left (right) ideal μ of M such that $\mu_t = A$.*

Proof. Let $\mu : M \to [0,1]$ be a fuzzy set defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

where t is a fixed number in $(0,1)$. Then clearly $\mu_t = A$. Let $x, y \in M$ and $\alpha \in \Gamma$. By routine calculations, we have that

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\}.$$

Now if $y \in A$, then $x\alpha y \in A$ because A is a left ideal of M. Hence $\mu(x\alpha y) = t = \mu(y)$. If $y \notin A$, then $\mu(y) = 0$ and so $\mu(x\alpha y) \geq \mu(y)$. Therefore μ is a fuzzy left ideal of M.

THEOREM 5. *Let μ be a fuzzy left (right) ideal of a Γ-ring M Then two level left (right) ideals μ_{t_1} and μ_{t_2} (with $t_1 < t_2$) of μ are equal if and only if there is no $x \in M$ such that $t_1 \leq \mu(x) < t_2$.*

Proof. ($\Rightarrow$) Suppose $t_1 < t_2$ and $\mu_{t_1} = \mu_{t_2}$. If there exists $x \in M$ such that $t_1 \leq \mu(x) < t_2$, then μ_{t_2} is a proper subset of μ_{t_1}. This is a contradiction.

($\Leftarrow$) Assume that there is no $x \in M$ such that $t_1 \leq \mu(x) < t_2$. From $t_1 < t_2$ it follows that $\mu_{t_2} \subseteq \mu_{t_1}$. If $x \in \mu_{t_1}$, then $\mu(x) \geq t_1$ and so $\mu(x) \geq t_2$ because $\mu(x) \not< t_2$. Hence $x \in \mu_{t_2}$. This completes the proof.

THEOREM 6. *Let M be a Γ-ring and μ a fuzzy left (right) ideal of M. If $Im(\mu) = \{t_1, ..., t_n\}$, where $t_1 < ... < t_n$, then the family of left (right) ideals $\mu_{t_i} (i = 1, ..., n)$ constitutes all the level left (right) ideals of μ.*

Proof. Let $t \in [0,1]$ and $t \notin Im(\mu)$. If $t < t_1$, then $\mu_{t_1} \subseteq \mu_t$. Since $\mu_{t_1} = M$, it follows that $\mu_t = M$, so that $\mu_t = \mu_{t_1}$. If $t_i < t < t_{i+1} (1 \leq i \leq n-1)$ then there is no $x \in M$ such that $t \leq \mu(x) < t_{i+1}$. From Theorem 5, we have that $\mu_t = \mu_{t_{i+1}}$. This shows that for any $t \in [0,1]$ with $t \leq \mu(0_M)$, the level left ideal μ_t is in $\{\mu_{t_i} | 1 \leq i \leq n\}$.

THEOREM 7. *Let A be a nonempty subset of a Γ-ring M and let μ be a fuzzy set in M such that μ is into $\{0,1\}$, so that μ is the characteristic function of A. Then μ is a fuzzy left (right) ideal of M if and only if A is a left (right) ideal of M.*

Proof. Assume that μ is a fuzzy left ideal of M. Let $x, y \in A$. Then $\mu(x) = \mu(y) = 1$. Thus $\mu(x - y) \geq \min\{\mu(x), \mu(y)\} = 1$ and so $\mu(x - y) = 1$. This means that $x - y \in A$. Therefore A is an additive subgroup of M. Let $x \in M, y \in A$ and $\alpha \in \Gamma$. Then $\mu(x\alpha y) \geq \mu(y) = 1$ and hence $\mu(x\alpha y) = 1$. So $x\alpha y \in A$, and A is a left ideal of M. The proof of converse is similar to that of Theorem 4.

DEFINITION 5. ([1]) Let M and N both be Γ-rings, and θ a mapping of M into N. Then θ is a Γ-homomorphism iff $\theta(x + y) = \theta(x) + \theta(y)$ and $\theta(x\alpha y) = \theta(x)\alpha\theta(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

DEFINITION 6. ([5]) If μ is a fuzzy set in M, and f is a function defined on M, then the fuzzy set ν in $f(M)$ defined by

$$\nu(y) = \sup_{x \in f^{-1}(y)} \mu(x)$$

for all $y \in f(M)$ is called the image of μ under f. Similarly, if ν is a fuzzy set in $f(M)$, then the fuzzy set $\mu = \nu \circ f$ in M (that is, the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in M$) is called the preimage of ν under f.

THEOREM 8. *A Γ-homomorphic preimage of a fuzzy left (right) ideal is a fuzzy left (right) ideal.*

Proof. Let $\theta : M \to N$ be a Γ-homomorphism of Γ-rings, ν a fuzzy left ideal of N and μ the preimage of ν under θ. Then

$$\begin{aligned}\mu(x - y) &= \nu(\theta(x - y)) \\ &= \nu(\theta(x) - \theta(y)) \\ &\geq \min\{\nu(\theta(x)), \nu(\theta(y))\} \\ &= \min\{\mu(x), \mu(y)\}\end{aligned}$$

and

$$\begin{aligned}\mu(x\alpha y) &= \nu(\theta(x\alpha y)) \\ &= \nu(\theta(x)\alpha\theta(y)) \\ &\geq \nu(\theta(y)) \\ &= \mu(y)\end{aligned}$$

for all $x, y \in M$ and $\alpha \in \Gamma$.

We say that a fuzzy set μ in M has the sup property ([5]) if, for any subset T of M, there exists $t_0 \in T$ such that

$$\mu(t_0) = \sup_{t \in T} \mu(t).$$

THEOREM 9. *A Γ-homomorphic image of a fuzzy left (right) ideal which has the sup property is a fuzzy left (right) ideal.*

Proof. Let $\theta : M \to N$ be a Γ-homomorphism of Γ-rings, μ a fuzzy left ideal of M with the sup property and ν the image of μ under θ. Given $\theta(x), \theta(y) \in \theta(M)$, let $x_0 \in \theta^{-1}(\theta(x))$, $y_0 \in \theta^{-1}(\theta(y))$ be such that

$$\mu(x_0) = \sup_{t \in \theta^{-1}(\theta(x))} \mu(t), \qquad \mu(y_0) = \sup_{t \in \theta^{-1}(\theta(y))} \mu(t),$$

respectively. Then

$$\begin{aligned}\nu(\theta(x) - \theta(y)) &= \sup_{z \in \theta^{-1}(\theta(x) - \theta(y))} \mu(z) \\ &\geq \mu(x_0 - y_0) \\ &\geq \min\{\mu(x_0), \mu(y_0)\} \\ &= \min\{\sup_{t \in \theta^{-1}(\theta(x))} \mu(t), \sup_{t \in \theta^{-1}(\theta(y))} \mu(t)\} \\ &= \min\{\nu(\theta(x)), \nu(\theta(y))\},\end{aligned}$$

and for any $\alpha \in \Gamma$,

$$\begin{aligned}\nu(\theta(x)\alpha\theta(y)) &= \sup_{z \in \theta^{-1}(\theta(x)\alpha\theta(y))} \mu(z) \\ &\geq \mu(x_0 \alpha y_0) \\ &\geq \mu(y_0) \\ &= \sup_{t \in \theta^{-1}(\theta(y))} \mu(t) \\ &= \nu(\theta(y)).\end{aligned}$$

This completes the proof.

References

1. W.E. Barnes, *On the Γ-rings of Nobusawa*, Pacific J. Math. **18** (1966), 411–422.
2. C.L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.
3. P.S. Das, *Fuzzy groups and level subgroups*, J. Math. Anal. Appl. **84** (1981), 264–269.
4. T.S. Ravisankar and U.S Shukla, *Structure of Γ-rings*, Pacific J Math. **80** (1979), 537–559.
5. A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.
6. L.A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338–353

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