Pusan Kyŏngnam Math. J. 8(1992), No. 2, pp. 117–127

TYPES OF $\sum_i \oplus \mathrm{Alg}\mathcal{L}_i$

JONG GEOUN KIM AND DAE YEON HA

1. Introduction

The study of non-self-adjoint operator algebras on Hilbert space was begun in 1974 by Arveson ([1]). Recently, such algebras have been found to be of use in physics, in electrical engineering, and in general systems theory. Of particular interest to mathematicians are reflexive algebras with commutative lattices of invariant subspaces. One of the most important classes of such algebras is the seguence $\mathrm{Alg}\mathcal{L}_2$, $\mathrm{Alg}\mathcal{L}_4, \cdots, \mathrm{Alg}\mathcal{L}_\infty$ of "tridiagonal" algebras, discovered by Gilfeather and Larson ([5]). We shall often disregard the distinction between an orthogonal projection and its range space. Let $\mathcal{L}$ be a family of orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then $\mathrm{Alg}\mathcal{L}$ is an algebra containing I (I represents the identity operator acting on $\mathcal{H}$) and $\mathrm{Alg}\mathcal{L}$ is closed in the weak operator topology.

In this paper, if $\sum_i \oplus \mathrm{Alg}\mathcal{L}_i$ is a von Neumann algebra, we want to find out what its type is.

We will introduce the terminologies which are used in the above general introduction and in the general theorems of this paper.

Let $\mathcal{C}$ be a subset of the class of all bounded operator acting on a Hilbert space $\mathcal{H}$. $\mathcal{C}$ is called self-adjoint if A^* is in $\mathcal{C}$ for every A in $\mathcal{C}$. If $\mathcal{C}$ is a vector space over $\mathbf{C}$ and if $\mathcal{C}$ is closed under the composition of maps, then $\mathcal{C}$ is called an algebra. $\mathcal{C}$ is called a self-adjoint algebra provided A^* is in $\mathcal{C}$ for every A in $\mathcal{C}$. Otherwise, $\mathcal{C}$ is called a non-self-adjoint algebra. $\mathcal{C}$ is a C^*-algebra if $\mathcal{C}$ is a self-adjoint algebra which is contains I and closed in the norm topology. $\mathcal{C}$ is a von Neumann algebra if $\mathcal{C}$ is a C^*-algebra which is closed in the weak operator topology.

For any subset $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, we shall denote by $\mathcal{A}'$ the commutant of $\mathcal{A}$:

$$\mathcal{A}' = \{B \in \mathcal{B}(\mathcal{H}) : BA = AB \text{ for any } A \text{ in } \mathcal{A}\}.$$

Received July 31, 1992.

For any subset $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, $\mathcal{A}'$ is an algebra which contains the identity operator I in $\mathcal{B}(\mathcal{H})$; moreover, it is easy to check that $\mathcal{A}'$ is closed in the strong operator topology (equivalently, it is closed in the weak operator topology). If $\mathcal{A}$ is self-adjoint, then $\mathcal{A}'$ is a von Neumann algebra. In particular, if $\mathcal{C}$ is a von Neumann algebra, then $\mathcal{C}'$ is a von Neumann algebra ([24]). Let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\mathcal{C}' \subset \mathcal{B}(\mathcal{H})$ its commutant. Then $\mathcal{C} \cap \mathcal{C}'$ is the common center of the algebras $\mathcal{C}$ and $\mathcal{C}'$. It is obvious that $\mathcal{C} \cap \mathcal{C}' \subset \mathcal{B}(\mathcal{H})$ is a (commutative) von Neumann algebra.

A von Neumann algebra is called a factor if its center is equal to the set of all saclar mutiples of the identity operator.

Let $\mathcal{H}$ be a complex Hilbert space. A linear manifold in $\mathcal{H}$ is a subset of $\mathcal{H}$ which is closed under vector addition and under multiplication by complex numbers. A subspace of $\mathcal{H}$ is a closed manifold.

We shall often disregard the distinction between an orthogonal projection and its range space. Thus we consider a subspace lattice as consisting of orthogonal projections or subspaces and we may use the same notation to indicate either. This occurs most often in the techinical arguments.

Let $\mathcal{L}$ be a subset of all orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then $\mathcal{L}$ is called a lattice if $\mathcal{L}$ is closed under the operators "$\wedge$" and "$\vee$" for finitely many elements of $\mathcal{L}$. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, Alg$\mathcal{L}$ denotes the algebra of all bounded operators acting on $\mathcal{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$, that is,

$$\text{Alg}\mathcal{L} = \{A \in \mathcal{B}(\mathcal{H}) : AE = EAE \text{ for any } E \text{ in } \mathcal{L}\}.$$

A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on a Hilbert space $\mathcal{H}$, containing $\mathbf{0}$ and I ($\mathbf{0}$ represents zero operator acting on $\mathcal{H}$). Dually, if $\mathcal{C}$ is a subalgebra of the set of all bounded operators acting on $\mathcal{H}$, then Lat$\mathcal{C}$ is the lattice of all orthogonal projections invariant for each operator in $\mathcal{C}$. An algebra $\mathcal{C}$ is reflexive if $\mathcal{C} =$ AlgLat$\mathcal{C}$. A lattice $\mathcal{L}$ is reflexive if $\mathcal{L} =$ LatAlg$\mathcal{L}$. A lattice $\mathcal{L}$ is commutative if each pair of orthogonal projections in $\mathcal{L}$ commutes. Especially, if $\mathcal{L}$ is a commutative subspace lattice, or

CSL, then Alg$\mathcal{L}$ is called a CSL algebra. Subspace lattices need not be reflexive ; however, commutative ones are reflexive ([5]). If $f_1, f_2, \cdots, f_n$ are vectors in some Hilbert space, then $[f_1, f_2, \cdots, f_n]$ means the subspace generated by the vectors $f_1, f_2, \cdots, f_n$.

Let $\mathcal{H}_i$ be a Hilbert space for each $i \in \mathcal{I}$. Then

$$\mathcal{H} = \{f = \{f_i\} : f_i \in \mathcal{H}_i \ \sum_i \|f_i\|^2 < \infty\}$$

is a Hilbert space when the algebraic structure, inner product, and norm are defined by

$$\{f_i\} + \{g_i\} = \{f_i + g_i\},$$
$$\|\{f_i\}\|^2 = \sum_i \|f_i\|^2.$$

The resulting Hilbert space $\mathcal{H}$ is called the (Hilbert) direct sum of $\mathcal{H}_1, \mathcal{H}_2, \cdots$, and is denoted by $\sum_i \oplus \mathcal{H}_i$.

Suppose that $\mathcal{H}_i$ is a Hilbert space, and $A_i \in \mathcal{B}(\mathcal{H})$ for each i. If $\sup\{\|A_i\| : i \in \mathcal{I}\} < \infty$, the equation $A\{f_i\} = \{A_i f_i\}$ defines a bounded operator A acting on $\sum_i \oplus \mathcal{H}_i$. We call A the direct sum $\sum_i \oplus A_i$ of the family $\{A_i\}$. We have

$$\|\sum_i \oplus A_i\| = \sup\{\|A_i\| : i \in \mathcal{I}\},$$
$$(\sum_i \oplus A_i)^* = \sum_i \oplus A_i^*,$$
$$\sum_i \oplus(\alpha A_i + \beta B_i) = \alpha(\sum_i \oplus A_i) + \beta(\sum_i \oplus B_i),$$
$$(\sum_i \oplus A_i)(\sum_i \oplus B_i) = \sum_i \oplus A_i B_i.$$

when $A_i, B_i \in \mathcal{B}(\mathcal{H}_i)$, and $\alpha, \beta \in \mathbf{C}$.

2. General Theorems

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$. $\mathcal{C}$ is a von Neumann algebra if $\mathcal{C}$ is a C^*-algebra which is closed in the weak operator topology. If $\mathcal{L}$ is a family of orthogonal projections acting on $\mathcal{H}$, then Alg$\mathcal{L}$ is an algebra containing I and closed in the weak operator topology. Therefore in order to prove that Alg$\mathcal{L}$ is a von Neumann algebra, it is sufficient to show that Alg$\mathcal{L}$ is self-adjoint.

THEOREM 1. *Let $\mathcal{L}$ be a family of orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then*
(a) Alg$\mathcal{L}$ *is an algebra containing I (I represents the identity operator acting on $\mathcal{H}$)*
(b) Alg$\mathcal{L}$ *is closed in the norm topology*
(c) Alg$\mathcal{L}$ *is closed in the weak operator topology.*

Proof. **(a)** Let A and B be in Alg$\mathcal{L}$. Then $AE = EAE$ and $BE = EBE$ for all E in $\mathcal{L}$. So $(A+B)E = AE + BE = EAE + EBE = E(A+B)E$ for all E in $\mathcal{L}$. Hence $A+B$ is in Alg$\mathcal{L}$. For every $\alpha \in \mathbb{C}$,

$$(\alpha A)E = \alpha(AE) = \alpha(EAE) = E(\alpha A)E.$$

Hence αA is in Alg$\mathcal{L}$, and

$$\begin{aligned}(AB)E &= A(BE) = A(EBE) = (AE)(BE) = (EAE)BE \\ &= EA(EBE) = (EA)(BE) = E(AB)E.\end{aligned}$$

Hence AB is in Alg$\mathcal{L}$. Thus Alg$\mathcal{L}$ is an algebra. For every E in $\mathcal{L}$, $E = E^2$ and so $IE = EIE$. Hence I is in Alg$\mathcal{L}$.
(b) Let $\{A_n\}$ be a sequence in Alg$\mathcal{L}$ and let A_n converge to A in the norm topology. Then for any E in $\mathcal{L}$ $\|A_nE - AE\| \leq \|A_n - A\|\|E\| = \|A_n - A\|$ and $\|EA_nE - EAE\| \leq \|E\|\|A_n - A\|\|E\| = \|A_n - A\|$. Hence A_nE and EA_nE converge to AE and EAE in the norm topology for any E in $\mathcal{L}$, respectively. Since $A_nE = EA_nE$ for all E in $\mathcal{L}$, we have $AE = EAE$. Therefore A is in Alg$\mathcal{L}$.
(c) Since Alg$\mathcal{L}$ is an algebra, it is convex. By [20], the weak and strong operator closures of Alg$\mathcal{L}$ coincide. So we have to show that Alg$\mathcal{L}$ is closed in the strong operator topology. Let $\{A_n\}$ be a sequence in Alg$\mathcal{L}$ and let A_n converge strongly to A. Then A_nf converges to Af for all

f in $\mathcal{H}$. Since $A_nE = EA_nE$ for all E in $\mathcal{L}$ and A_nEf converges to AEf for all f in $\mathcal{H}$, EA_nEf converges to $EAEf$ for all f in $\mathcal{H}$. Hence $AEf = EAEf$ for all f in $\mathcal{H}$ and hence $AE = EAE$ for all E in $\mathcal{L}$. So A is in Alg$\mathcal{L}$. That is, Alg$\mathcal{L}$ is closed in the strong operator topology.

LEMMA 2. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be families of orthogonal projections acting on a Hilbert space $\mathcal{H}$. If $\mathcal{L}_1 \subset \mathcal{L}_2$, then* $\mathrm{Alg}\mathcal{L}_2 \subset \mathrm{Alg}\mathcal{L}_1$.

Proof. Let A be in Alg$\mathcal{L}_2$. Then $AE = EAE$ for all E in $\mathcal{L}_2$. Since $\mathcal{L}_1 \subset \mathcal{L}_2$, $AE = EAE$ for all E in $\mathcal{L}_1$. Hence A is in Alg$\mathcal{L}_1$.

Let E and F be orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then a partial order relation $\leq$ is described as follows : $E \leq F$ if and only if $EF = FE = E$. E, F are said to be mutually orthogonal if $EF = 0$.

LEMMA 3. *Let $\mathcal{L}$ be a lattice of orthogonal projections acting on a Hilbert space $\mathcal{H}$ and let $\mathcal{F}$ be a family of mutually orthogonal projections acting on $\mathcal{H}$. If $\mathcal{L}$ is the lattice generated by $\mathcal{F}$, then* $\mathrm{Alg}\mathcal{L} = \mathrm{Alg}\mathcal{F}$.

Proof. By Lemma 2, we shall show that Alg$\mathcal{F}$ is included in Alg$\mathcal{L}$. Let A be an element in Alg$\mathcal{F}$ and let E be in $\mathcal{L}$. Since $\mathcal{F}$ is the family of mutually orthogonal projections acting on $\mathcal{H}$, there exists F_i in $\mathcal{F}$, $i = 1, 2, \cdots, n$ such that $E = \vee_{i=1}^n F_i$. Hence

$$\begin{aligned}
EAE &= (\vee_{i=1}^n F_i)A(\vee_{i=1}^n F_i) = (\sum_{i=1}^n F_i)A(\sum_{i=1}^n F_i) \\
&= F_1A(\sum_{i=1}^n F_i) + F_2A(\sum_{i=1}^n F_i) + \cdots + F_nA(\sum_{i=1}^n F_i) \\
&= F_1AF_1 + F_2AF_2 + \cdots + F_nAF_n \\
&= AF_1 + AF_2 + \cdots + AF_n \\
&= A(\sum_{i=1}^n F_i) = A(\vee_{i=1}^n F_i) = AE.
\end{aligned}$$

Thus A is in Alg$\mathcal{L}$.

THEOREM 4. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{F}$ be a family of mutually orthogonal projections acting on $\mathcal{H}$ such that $\vee\mathcal{F} = I$. If $\mathcal{L}$ is the lattice generated by $\mathcal{F}$, then* Alg$\mathcal{L}$ *is a von Neumann algebra.*

Proof. From Theorem 1, Alg$\mathcal{L}$ is an algebra containing I and closed in the weak operator topology. Therefore it is sufficient to show that Alg$\mathcal{L}$ is self- adjoint. Let A be an element in Alg$\mathcal{L}$. Suppose that $\mathcal{F} = \{ E_1, E_2, \cdots \}$, where E_i is an orthogonal projection acting on $\mathcal{H}$ for all $i = 1, 2, \cdots$. Since A is in Alg$\mathcal{L}$, $AE_i = E_iAE_i$ for all $i = 1, 2, \cdots$. By $AE_i^{\perp} = E_i^{\perp}AE_i^{\perp}$ for all $i = 1, 2, \cdots$ and hence $E_i^{\perp}A^* = E_i^{\perp}A^*E_i^{\perp}$ for all $i = 1, 2, \cdots$. Since $E_i^{\perp} = I - E_i$ for each $i = 1, 2, \cdots$,

$$\begin{aligned} E_i^{\perp}A^* = A^* - E_iA^* &= (I - E_i)A^*(I - E_i) \\ &= A^* - E_iA^* - A^*E_i + E_iA^*E_i. \end{aligned}$$

Hence $A^*E_i = E_iA^*E_i$ for all $i = 1, 2, \cdots$. Therefore by Lemma 3, A^* is in Alg$\mathcal{L}$, i.e. Alg$\mathcal{L}$ is self-adjoint.

THEOREM 5. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{F}$ be a mutually orthogonal family of closed subspaces of $\mathcal{H}$ and let $\mathcal{L}$ be the lattice generated by $\mathcal{F}$. If $\vee\mathcal{F} \neq \mathcal{H}$, then* Alg$\mathcal{L}$ *is not a von Neumann algebra.*

Proof. Suppose that $\mathcal{F} = \{\mathcal{H}_1, \mathcal{H}_2, \cdots\}$, where $\mathcal{H}_i$ is a closed subspace of $\mathcal{H}$ for all $i = 1, 2, \cdots$. Let A be in Alg$\mathcal{L}$. Then A is in Alg$\mathcal{F}$ by Lemma 3. Hence A has the following matrix form on $\sum_i \oplus\mathcal{H}_i$:

$$(*) \qquad \begin{pmatrix} \overbrace{\begin{matrix} \overbrace{A_{11}}^{\mathcal{H}_1} & \overbrace{\qquad}^{\mathcal{H}_2} & \overbrace{\qquad}^{\mathcal{H}_3} & \overbrace{\qquad}^{\cdots} \end{matrix}}^{\vee\mathcal{F}} & \overbrace{\qquad}^{(\vee\mathcal{F})^{\perp}} \\ \begin{matrix} & A_{22} & & 0 \\ & & A_{33} & \\ & 0 & & \ddots \end{matrix} & B \\ 0 & C \end{pmatrix},$$

where $A_{ii} : \mathcal{H}_i \to \mathcal{H}_i$ is an operator such that $A_{ii} = A|_{\mathcal{H}_i}$ for all $i = 1, 2, \cdots$, B and C are operators from $(\vee\mathcal{F})^{\perp}$ into $\vee\mathcal{F}$ and $(\vee\mathcal{F})^{\perp}$, respectively and all other entries are $\mathbf{0}$. In particular, we can take an operator A_0 in Alg$\mathcal{L}$ which has a nonzero operator $B_0 : (\vee\mathcal{F})^{\perp} \to \vee\mathcal{F}$ in the matrix from $(*)$. Therefore Alg$\mathcal{L}$ is not self-adjoint. Thus Alg$\mathcal{L}$ is not a von Neumann algebra.

THEOREM 6. ([24]) *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a $*$-algebra of operators with $I \in \mathcal{C}$. Then the following statements are equivalent.*
(a) *$\mathcal{C}$ is a von Neumann algebra.*
(b) *$\mathcal{C} = \mathcal{C}''$, where $\mathcal{C}''$ is the bicommutant of $\mathcal{C}$.*

THEOREM 7. *Let $\mathcal{F}_i$ be a family of orthogonal projections acting on a separable Hilbert space $\mathcal{H}_i$ such that $\bigvee \mathcal{F}_i = I$ for each i. If $\mathcal{L}_i$ is the lattice generated by $\mathcal{F}_i$ for each i, then $\sum_i \oplus \mathrm{Alg}\mathcal{L}_i = \{A = \sum_i \oplus A_i : A_i \in \mathrm{Alg}\mathcal{L}_i, \sup_i\{\|A_i\|\} < \infty\}$ is a von Neumann algebra.*

Proof. Let $\mathcal{H} = \sum_i \oplus \mathcal{H}_i$. Let $E_i \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection onto $\mathcal{H}_i$, and let $\mathcal{D} = \{B = \sum_i \oplus B_i : B_i \in (\mathrm{Alg}\mathcal{L}_i)', \sup_i\{\|B_i\|\} < \infty\}$. Put $\mathcal{A} = \sum_i \oplus \mathrm{Alg}\mathcal{L}_i$. Then $\mathcal{A}$ and $\mathcal{D}$ are $*$-algebras of operators acting on $\mathcal{H}$, and $AB = BA$, for each $A \in \mathcal{A}$ and $B \in \mathcal{D}$ (i.e., $\mathcal{A} \subset \mathcal{D}'$ or equivalently, $\mathcal{D} \subset \mathcal{A}'$). Now suppose that $T \in \mathcal{B}(\mathcal{H})$ commutes with each operator in $\mathcal{D}$. Since E_i is in $\mathcal{D}$, $TE_i = E_iT$, for each i, and if $T_i = T\,|_{\mathcal{H}_i}$, then $T_i \in \mathcal{B}(\mathcal{H}_i)$ and $T = \sum_i \oplus T_i$. Hence if $B \in \mathcal{D}$, $B = \sum_i \oplus B_i$ with each $B_i \in (\mathrm{Alg}\mathcal{L}_i)'$, then $\sum_i \oplus T_iB_i = TB = BT = \sum_i \oplus B_iT_i$ So $T_iB_i = B_iT_i$, for each $B_i \in (\mathrm{Alg}\mathcal{L}_i)'$, i.e., $T_i \in (\mathrm{Alg}\mathcal{L}_i)'' = \mathrm{Alg}\mathcal{L}_i$. Thus $T \in \mathcal{A}$, so $\mathcal{D}' = \mathcal{A}$. Therefore $\mathcal{A} = \mathcal{A}''$. By Theorem 6, $\mathcal{A} = \sum_i \oplus \mathrm{Alg}\mathcal{L}_i$ is a von Neumann algebra.

Interchanging the roles of Alg$\mathcal{L}_i$ and $(\mathrm{Alg}\mathcal{L}_i)'$, we obtain $\mathcal{A}' = \mathcal{D}$. Hence we can get the following corollary.

COROLLARY 8 *Let $\mathcal{F}_i$ and $\mathcal{L}_i$ be families as defined in* Theorem 7. *If $\mathcal{A} = \sum_i \oplus \mathrm{Alg}\mathcal{L}_i$, then $\mathcal{A}' = \sum_i \oplus (\mathrm{Alg}\mathcal{L}_i)'$.*

From Lemma 2, we can get the following lemma.

LEMMA 9. *Let $\mathcal{F}_i$ and $\mathcal{L}_i$ be families of orthogonal projections acting on a Hilbert space $\mathcal{H}_i$ for each i. If $\mathcal{F}_i \subset \mathcal{L}_i$ for each i, then $\sum_i \oplus \mathrm{Alg}\mathcal{L}_i \subset \sum_i \oplus \mathrm{Alg}\mathcal{F}_i$.*

From Lemma 3, we can get the following lemma.

LEMMA 10. *Let $\mathcal{L}_i$ be a lattice of orthogonal projections acting on a Hilbert space $\mathcal{H}_i$ for each i and let $\mathcal{F}_i$ be a family of mutually orthogonal projections acting on $\mathcal{H}_i$ for each i. If $\mathcal{L}_i$ is the lattice generated by $\mathcal{F}_i$ for each i, then $\sum_i \oplus \mathrm{Alg}\mathcal{L}_i = \sum_i \oplus \mathrm{Alg}\mathcal{F}_i$.*

DEFINITION 11. Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let $\mathcal{P}_\mathcal{C}$ be the set of orthogonal projections in $\mathcal{C}$.
(a) Two orthogonal projections E, F in $\mathcal{P}_\mathcal{C}$ are said to be equivalent, and this relation is denoted by $E \sim F$, if there exists a partial isometry U in $\mathcal{C}$ such that $E = U^*U$ and $F = UU^*$; then $UE = U = FU$. We say that E is dominated by F, and we denote by $E \prec F$ this relation, if E is equivalent to a subprojection of F.
(b) An orthogonal projection E in $\mathcal{P}_\mathcal{C}$ is said to be abelian if $E\mathcal{C}E$ is commutative.
(c) An orthogonal projection E in $\mathcal{P}_\mathcal{C}$ is said to be finite if whenever $E \sim F \leq E$ for an orthogonal projection F in $\mathcal{P}_\mathcal{C}$, it follows that $F = E$.
(d) An orthogonal projection E in $\mathcal{P}_\mathcal{C}$ is said to be a central projection if belongs to the center $\mathcal{C} \cap \mathcal{C}'$ of $\mathcal{C}$.
(e) An orthogonal projection E in $\mathcal{P}_\mathcal{C}$ is said to be properly infinite if whenever PE is finite, for each central projection P in $\mathcal{P}_\mathcal{C}$, it follows that $PE = 0$.

DEFINITION 12. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra.
(a) $\mathcal{C}$ is said to be finite if I is a finite orthogonal projection.
(b) $\mathcal{C}$ is said to be semifinite if any nonzero central projection contains a nonzero finite orthgonal projection.
(c) $\mathcal{C}$ is said to be of type I if any nonzero central projection contains a nonzoro abelian orthogonal projection.
(d) $\mathcal{C}$ is said to be of type II if it is semifinite and it does not contain any nonzero abelian orthogonal projection.
(e) $\mathcal{C}$ is said to be of type III if it dose not contain any nonzero finite orthogonal projection.
(f) $\mathcal{C}$ is said to be type I_{fin} if it is finite and of type I.
(g) $\mathcal{C}$ is said to be type I_∞ if it not finite and it is of type I.
(h) $\mathcal{C}$ is said to be of type II_1 if it is finite and of II.
(i) $\mathcal{C}$ is said to be of type II_∞ if it is not finite, but it is of type II.

LEMMA 13. ([25]) *Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{B}(\mathcal{H})$ is of type I.*

LEMMA 14. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra.Then $\mathcal{C}$ is finite if and only if* $\dim \mathcal{H} < \infty$.

EXAMPLE 15. Let $\mathcal{H}$ be a separable infinite Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots\}$ and let $\mathcal{F} = \{[e_i] : i = 1, 2, \cdots\}$. If $\mathcal{L}$ is the lattice generated by $\mathcal{F}$, then Alg$\mathcal{L}$ is of type I_∞.

THREOREM 16. *Let $\mathcal{H}$ be a separable infinite Hilbert space and let $\mathcal{F}$ be a family of mutually orthogonal projection acting on $\mathcal{H}$ such that $\vee\mathcal{F} = I$. If $\mathcal{L}$ is the lattice generated by $\mathcal{F}$, then* Alg$\mathcal{L}$ *is of type I_∞.*

Proof. Suppose that $\mathcal{F} = \{E_1, E_2, \cdots\}$ and $\mathcal{H}_i$ is the closed subspace of $\mathcal{H}$ such that $E_i(\mathcal{H}) = \mathcal{H}_i$ for all $i = 1, 2, \cdots$. Let A be in Alg$\mathcal{L}$. Since Alg$\mathcal{L}$ = Alg$\mathcal{F}$ by Lemma 3, A is in Alg$\mathcal{F}$. Hence A has the following matrix form on $\sum_i \oplus \mathcal{H}_i$:

$$\begin{pmatrix} \overbrace{A_{11}}^{\mathcal{H}_1} & \overbrace{\phantom{A_{22}}}^{\mathcal{H}_2} & \overbrace{\phantom{A_{33}}}^{\mathcal{H}_3} & \overbrace{}^{\cdots} \\ & A_{22} & & 0 \\ & & A_{33} & \\ & 0 & & \ddots \end{pmatrix},$$

where $A_{ii} : \mathcal{H}_i \to \mathcal{H}_i$ is the operator such that $A_{ii} = A\mid_{\mathcal{H}_i}$ for all $i = 1, 2, \cdots$. Let B be in $(Alg\mathcal{L})'$ and let $B_{ij} : \mathcal{H}_j \to \mathcal{H}_i$ be the operator such that $B_{ij} = E_i B\mid_{\mathcal{H}_j}$ for all $i, j = 1, 2, \cdots$. Since $AB = BA$ for all A in Alg$\mathcal{L}$, $A_{ii}B_{ii} = B_{ii}A_{ii}$ for all $i = 1, 2, \cdots$, and $B_{ij} = \mathbf{0}$ ($i \neq j$; $i, j = 1, 2, \cdots$). So

$$P = \begin{pmatrix} P_{11} & 0 & 0 & 0 & \cdots \\ 0 & P_{22} & 0 & 0 & \cdots \\ 0 & 0 & P_{33} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

is a nonzero central projection in Alg$\mathcal{L}$ if and only if P_{ii} is an orthogonal projection acting on $\mathcal{H}_i$ for all $i = 1, 2, \cdot$ and P_{kk} is not zero for some k. If P_{kk} is a nonzero orthogonal projection acting on $\mathcal{H}_k$ for some k,

P_{kk} contains a subprojection F_{kk} of rank one. Let F be the orthogonal projection acting on $\sum_i \oplus \mathcal{H}_i$ such that $E_k F|_{\mathcal{H}_k} = F_{kk}$ and $E_i F|_{\mathcal{H}_j} = 0$ if $i \neq k$ or $j \neq k$. Then F is Alg$\mathcal{L}$ and F is a nonzero abelian subprojection of P. Hence Alg$\mathcal{L}$ is of type I. By Lemma 14, Alg$\mathcal{L}$ is of type I_∞.

THEOREM 17. *Let $\mathcal{F}_i$ be a family of mutually orthogonal projections acting on a separable Hilbert space $\mathcal{H}_i$ such that $\vee\mathcal{F}_i = I$ for each i. If $\mathcal{L}_i$ is the lattice generated by $\mathcal{F}_i$ for each i, then $\sum_i \oplus$Alg$\mathcal{L}_i$ is of type I_∞.*

Proof. Let P be a nonzero central orthogonal projection in $\sum_i \oplus$ Alg$\mathcal{L}_i$ for each i. Then $P = \sum_i \oplus P_i$, where P_i is in Alg$\mathcal{L}_i$ for each i. Since Alg$\mathcal{L}_i$ is of type I_∞ for each i by Theorem 16, there exist a nonzero abelian subprojection F_i of P_i in Alg$\mathcal{L}_i$ for each i. Hence $\sum_i \oplus F_i$ is a nonzero abelian subprojection of P in $\sum_i \oplus$ Alg$\mathcal{L}_i$. Therefore $\sum_i \oplus$ Alg$\mathcal{L}_i$ is of type I_∞.

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Department of Basic Studies
Pusan Junior College
Pusan 616-092, Korea

Department of Basic Studies
Ulsan Junior College
Uisan 680–190, Korea