THE JOURNAL OF THE KOREAN ASTRONOMICAL SOCIETY
25: 1 ~ 9, 1992

VELOCITY ANALYSIS OF M13 BY MAXIMUM LIKELIHOOD METHOD*

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(Received May 10, 1992)

ABSTRACT

We present new approach to analysis of velocity data of globular clusters. Maximum likelihood method is applied to get model parameters such as central potential, anisotropy radius, and total mass fractions in each mass class. This method can avoid problems in conventional binning method of chi-square. We utilize three velocity components, one from line of sight radial velocity and two from proper motion data. In our simplified scheme we adopt 3 mass-component model with unseen high mass stars, intermediate visible stars, and low mass dark remnants. Likelihood values are obtained for 124 stars in M13 for various model parameters. Our preferred model shows central potential of $W_o = 7$ and anisotropy radius with 7 core radius. And it suggests non-negligible amount of unseen high mass stars and considerable amount of dark remnants in M13.

I. INTRODUCTION

Many models for kinematic data of globular clusters were introduced. First theoretically based one is Michie (1963) type, which uses lowered Maxwellian distribution functions for anisotropic velocity dispersions. Then King (1966) adopted simpler scheme for isotropic case. King's model uses only one mass component, whereas Michie model can use multi-mass components.

Analysis of globular cluster data with these models have been published by many people (Da Costa & Freeman 1976 ; Gunn & Griffin 1979). Luminosity profile, which is basically the position information, can constrain the model parameters in King-Michie

* This work is supported by KOSEF under grant 913-0208-004-1

type distribution. However luminosity profile is not sensitive to the anisotropy radius. So we need to consider velocity data which can constrain anisotropy radius better. Velocity data also well represent the reflection of whole potential, whereas luminosity profile only represent visible stars. The possibility of combination of anisotropy and rotation effect were demonstrated by Lupton and Gunn (1986). And nice application of this method was done on M13 (Lupton et al. 1987). They used radial velocity data to constrain their model parameters. However their best model couldn't fit the radial velocity data successfully. It is probably because there are intrinsic problems of binning method such as bin boundary and bin size. In our approach we can avoid these problems if we use maximum likelihood method (Cudworth et al. 1984 ; Oh 1986), which utilize the individual velocity data rather than binning.

In our simplified model we do not consider the effect of rotation. However it is not too bad approximation for the case of M13, if we consider the large dispersion in rotation data.

II. BASIC SCHEME OF MAXIMUM LIKELIHOOD METHOD

The conventional method is based on minimizing the chi-square of binned data. The binning method has ambiguities such as bin size and bin boundaries. This approach is based on the assumption that the data have a Gaussian distribution in phase space, which is inconsistent with the models to be fitted. And there is no rigorous way to separate measurement error from the derived dispersion velocities. Some of these problems can be overcome if the velocity data and the star count data are analyzed with a maximum likelihood method. The basic principle of this method is to fit the velocities of every member star and the star count data directly to the stellar distribution function of a theoretical model such as the King-Michie model. For any given theoretical model, the stellar distribution function is determined by a set of structural parameters such as the central potential, anisotropy radius and stellar mass function. When it is properly normalized, the six dimensional phase space distribution function can be used as a probability function for the individual stars with particular values of velocity and position. For a given set of structural parameters, the total probability of the model can be evaluated from the product of the individual probability functions. In reality the stellar position and the velocity data are measured in terms of the projected distance from the cluster center. But the maximum likelihood method can still be used after the integration over the missing phase space elements. The measurement error can be accounted for rigorously by deconvolution.

III. ASTROPHYSICAL DATA

1. Velocity Data

Proper motion data contains two components of velocity of a star. And radial velocity, i.e. line of sight velocity, is the 3rd component. Cudworth and Monet (1978) measured relative proper motions for some 350 stars with over 90 % probabilities. Because it gives small proper motion at the distance of M13, maximum measurement errors are about ±13 mas/century which is equivalent to $\sim$ 4 km s^{-1}. It is quite large amount comparing with central velocity dispersion of $\sim$ 7 km s^{-1}. But we can still utilize proper motion data because the errors are deconvolved in our analysis. Lupton et al. (1987) did very good job to measure radial velocities for 154 stars in M13 with typical measurement error of 1 km s^{-1}. We picked up 124 stars which are common in both Cudworth and Monet's (1978) proper motion data and Lupton et al.'s (1987) radial velocity data. Even though we have all 3 components in velocity, still one position component is missing. We can manage this problem by integration along the direction of missing component.

2. Distance Modulus

We used the distance modulus of 14.07 derived from the proper motion data by Lupton et al. (1987).

3. Mass Function

Lupton et al. (1987) used very delicate mass function with more than 10 mass components. However we adopted 3 mass components in our simplfied model. If we consider uncertainties in conversions from observed luminosity to mass function, amount of unseen high mass and low mass stars, then our approximation is not too bad.

4. Rotation

Rotation effect is known for M13 by Lupton et al. (1987). They tried to match the rotation data by their three integral model. However the dispersion in rotation data is too big (up to 2 km s^{-1}) and also has a zigzag shape. So it is doubtful whether their best model succeeded to match the data. So we ignore the rotation effect in our simplified model.

IV. MODELS

The distribution function for multi-mass anisotropic case (Gunn & Griffin 1979) is

$$f_j(E,J) = e^{-\beta J^2}(e^{-A_j E} - 1) \tag{1}$$

for j-th mass class, where $J, E, \beta, and A_j$ are angular momentum, energy, and constants respecitively. If we measure velocities and radii in terms of the central velocity dispersion and core radius respectively and scale the potential by the central velocity dispersion, then eq. (1) can be converted into dimensionless form,

$$f_j(\xi,u) = \alpha_j C_j exp(-\frac{1}{2}\mu_j u_\perp^2 \frac{\xi^2}{\xi_a^2}) \left[exp(-\frac{1}{2}\mu_j u^2 + \mu_j W) - 1 \right], \tag{2}$$

where ξ and u are the dimensionless distance and dimensionless velocity, and $\alpha_j = \rho_{oj}/\rho_o$ is the fractional density contribution of mass class j at the center of the cluster, $\mu_j = m_j/\bar{m} is weighted mass of j-th mass class wrt mean mass, \xi_a$ is the anisotropy radius. Distribution function is basically a density in 6-dimensional phase space. So we can define the probability between V and $V + dV$ (where V is the 6-dimensional volume) as

$$P_{ji}dV = P_{ji}d^3\xi d^3u = \frac{f_{ji}(\xi, u)d^3\xi d^3u}{M_j}, \tag{3}$$

where normalization factor, $M_j = \int_\xi \int_u f_j(\xi, u)d^3\xi d^3u$, is the total mass. Therefore probability for all stars in j-th mass class is

$$P_j = \frac{f_j}{M_j} = \sum_i P_{ji} = \frac{\sum_i f_{ji}}{M_j}, \tag{4}$$

and i is the summation index for individual star. For a single star in j-th mass class the probability, P_{ji}, is

$$P_{ji} = \frac{f_{ji}}{M_j}. \tag{5}$$

After integrating along the line of sight, the distribution function is

$$f_{ji} = 2\alpha_j C_j \int_0^L exp(-\frac{\mu_j}{2}u_{\perp i}^2 \frac{\xi_i^2}{\xi_a^2}) \left[exp(-\frac{\mu_j}{2}u_i^2 + \mu_j W) - 1\right] dl, \tag{6}$$

where $L = [r_{lim}^2 - R_i^2]^{\frac{1}{2}}$, r_{lim} is the cutoff radius of the model and R_i is the projected distance of i-th star from the center of the cluster. We set the positive $x-$ axis toward observer, so $y - z$ plane is the projected plane. For the convenience we transformed the velocity data onto $x - y$ plane.

Next we consider the measurement error which is basically Gaussian distribution. We adopt deconvolution method, e.g. for x component of arbitrary function $g(x')$,

$$\frac{g(x) = \int g(x')exp\left[-\frac{(x'-x)^2}{2\sigma_x^2}\right] dx'}{\int exp\left[-\frac{(x'-x)^2}{2\sigma_x^2}\right] dx'.} \tag{7}$$

It is possible to integrate wrt u_z' analytically, therefore probability of i-th star in j-th mass class after error convolution,

$$P_{ji} = \frac{2\alpha_j C_j}{M_j} \int_0^L \frac{J_1(u_x, u_y, u_z)}{J_2(u_x, u_y, u_z)} dl \tag{8}$$

where

$$J_1(u_x,u_y,u_z) = A_1 \int_{-C_1}^{C_1}\int_{-C_2}^{C_2} A_2C_3[A_3A_4\{erf(A_5+A_6)-erf(A_5-A_6)\}- A_7\{erf(A_8+A_9)-erf(A_8-A_9)\}]du'_x du'_y, \tag{9}$$

$$J_2(u_x,u_y,u_z) = B_1 \int_{-C_1}^{C_1}\int_{-C_2}^{C_2} C_3\{erf(B_2+B_3)-erf(B_2-B_3)\}du'_x du'_y, \tag{10}$$

and constants are

$$\begin{aligned}
A_1 =& \frac{\sqrt{\pi}}{2},\\
A_2 =& exp\left[-\frac{\mu_j\xi^2}{2\xi_a^2}\left(-\frac{R_i}{\xi}u'_x + \frac{\sqrt{\xi^2-R_i^2}}{\xi}u'_y\right)^2\right],\\
A_3 =& exp\ \frac{\mu_j}{2}D_1,\\
A_4 =& \frac{1}{\sqrt{D_3}}exp\left(\frac{D_4^2}{D_3}-D_5\right),\\
A_5 =& \frac{D_4}{\sqrt{D_3}},\\
A_6 =& \sqrt{D_1D_3},\\
A_7 =& \frac{1}{\sqrt{D_2}}exp\left(\frac{D_4^2}{D_2}-D_5\right),\\
A_8 =& \frac{D_4}{\sqrt{D_2}},\\
A_9 =& \sqrt{D_1D_2},\\
B_1 =& \sqrt{2}A_1\sigma_z,\\
B_2 =& -\frac{u_z}{\sqrt{2}\sigma_z},\\
B_3 =& \frac{\sqrt{D_1}}{\sqrt{2}\sigma_z},\\
C_1 =& \sqrt{2W},\\
C_2 =& \sqrt{2W-u'^2_y},\\
C_3 =& exp\left[-\frac{(u'_x-u_x)^2}{2\sigma_x^2}-\frac{(u'_y-u_y)^2}{2\sigma_y^2}\right].
\end{aligned}$$

$$
\begin{aligned}
D_1 &= 2W - u_x'^2 - u_y'^2, \\
D_2 &= \frac{\mu_j}{2}\frac{\xi^2}{\xi_a^2} + \frac{1}{2\sigma_z^2}, \\
D_3 &= D_2 + \frac{\mu_j}{2}, \\
D_4 &= -\frac{u_z}{2\sigma_z^2}, \\
D_5 &= -u_z D_4.
\end{aligned}
$$

The observed quantities are $u_x, u_y, u_z, R_i, \sigma_x, \sigma_y, \sigma_z$. It turns out probability (eq. (8)) is expressed in triple integration after integrating analytically once. Then we integrate numerically.

Likelihood value for j-th mass class can be obtained as

$$L_j = ln \prod_i P_{ji}. \tag{11}$$

In summary for given parameters for King-Michie distribution function (eq. (1)) we can get the radial profile of potential (W) by solving the Poisson equation. From this profile we can get the likelihood value. Then we can find model parameters which maximize the likelihood value by varying numbers.

V. FITS

In our method we try to find model parameters in dimensionless space. We need to know the physical parameters such as core radius and central velocity dispersions. It seems that the best model for M13 so far is that of Lupton et al. (1987). So we adopt values from their model f, core radius of 1.60 pc, central velocity dispersion of 7.09 km s^{-1}, and distance of 6.5 kpc. Those values do not depend on model parameters in our analysis because we use dimensionless parameters. They also used more than 10 mass components. In their best model mass function showed a peak at 0.65 $M_\odot$. And they chose 1.2 $M_\odot$ for high mass end and 0.33 $M_\odot$ for cutoff. So we adopted 3 mass component model with 1.2 $M_\odot$, 0.65 $M_\odot$, 0.33 $M_\odot$. Most of stars with observed velocity data are giants. So 0.65 $M_\odot$ stars represents visible stars, 1.2 $M_\odot$ stars for unseen white dwarfs and 0.33 $M_\odot$ for dark remnants.

We calulate 79 models with various model parameters. The values of 12 models are shown in Table 1. The first column shows the model names. The central potential in the units of central velocity dispersion (W_o) of each model is given in column 2. Column 3 lists the anisotropy radius (ξ_a) in units of core radius. The total mass fractions in each mass class (M_1, M_2, M_3) are shown in columns 4, 5, and 6. The fractional density contributions ($\alpha_1, \alpha_2, \alpha_3$) of each mass class are given in columns 7, 8, and 9. The cutoff radius (ξ_t) in units of core radius is in column 10. Finally column 11 shows the likelihood value in units of 10^3.

Table 1. Model Parameters and Likelihood Values

Model (1)	W_o (2)	ξ_a (3)	M_1 (4)	M_2 (5)	M_3 (6)	α_1 (7)	α_2 (8)	α_3 (9)	ξ_t (10)	$L(10^3)$ (11)
a	7	6	0.15	0.67	0.18	0.5	0.45	0.05	22.6	-1.001
b	7	6	0.14	0.53	0.33	0.5	0.4	0.1	20.1	-1.000
c	7	7	0.14	0.54	0.32	0.5	0.4	0.1	19.3	-0.997
d	7	6	0.12	0.33	0.55	0.5	0.3	0.2	16.8	-1.002
e	7	7	0.09	0.60	0.31	0.4	0.5	0.1	19.7	-1.009
f	8	6	0.06	0.54	0.40	0.4	0.5	0.1	31.8	-1.037
g	7	5	0.09	0.58	0.33	0.4	0.5	0.1	22.1	-1.017
h	8	6	0.10	0.50	0.40	0.5	0.4	0.1	29.7	-1.026
i	7	5	0.13	0.53	0.34	0.5	0.4	0.1	21.5	-1.005
j	10	5	0.01	0.20	0.79	0.2	0.6	0.2	20.5	-1.091
k	10	7	0.01	0.23	0.76	0.2	0.6	0.2	78.9	-1.086
l	6	6	0.18	0.55	0.27	0.5	0.4	0.1	13.9	-0.958

One star beyond cutoff radius is skipped in model l.

Model j is presented because it has same central potential and anisotropy radius as Lupton et al.'s (1987). But it does not give maximum values. Moreover model k with larger anisotropy has larger likelihood value, which is opposite trend to the result of Lupton et al.'s values of $\xi_a = 3 \sim 5$. Models a - i and l are investigated to find best case. Model a, b, c, and d give similar likelihood values, anisotropy radius of $6 \sim 7$ core radius, and central potential of 7. This centarl potential is much different from Lupton et al.'s value of 10. Also amount of unseen high mass stars is not negligible and there is significant amount of low mass dark remnants. Maximum value of likelihood value is given in model l. However one star is beyond the cutoff radius, so this star is skipped in calculating likelihood value. Lupton et al.'s (1987) star count data show the cutoff radius of M13 is about 20 core radius. It turns out models with larger anisotropy radius than that of model c give smaller cutoff radius. So the models with smaller than about 20 core radius are limited in our analysis. Finally model c is chosen as our best model. The absolute value of 124 velocity data are plotted in Fig. 1. This is not velocity dispersion, but they are closely related to velocity dispersions because tyey are evaluated wrt mean velocities. We also plot theoretical velocity dispersions derived from model c after integrating along the line of sight direction to account for the projection effect like the observed velocity data. Individual data are shown rather than binned quantity because we utilize each data in our analysis.

VI. DISCUSSION

We analyzed the velocity data of 124 stars in M13 with maximum likelihood method,

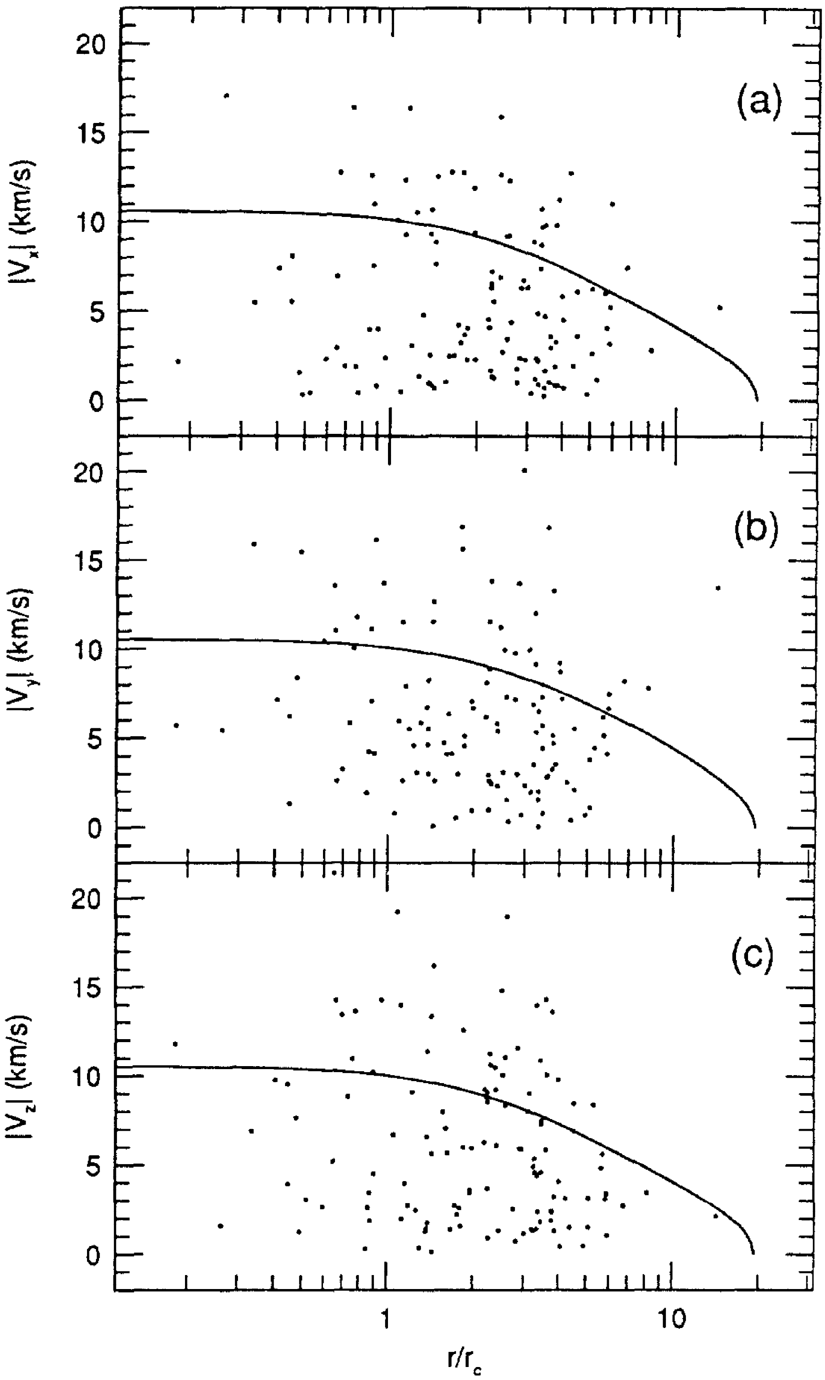

Fig. 1. The 124 velocity data of M13 and velocity dispersion from best model c are shown. Dots are velocity data and solid lines are theoretical velocity dispersions, (a) ; for line of sight direction, (b) & (c) ; radial and transverse direction on projected plane.

which can avoid problems in conventional chi-square method. We explore model parameters in dimensionless space. Those dimensionless parameters are central potential, anisotropy radius, total mass fractions in each mass class. We adopt 3 mass-component and non-rotating model in our simplified scheme. Our preferred model c gives quite different values from those of Lupton et al. (1987). It is not certain at this moment whether it is due to our simplified scheme. Lupton et al. (1987) derived their dimensionless model parameters mainly from star count data, whereas our results are purely from velocity informations. So our approach will constrain anisotropy radius better, because it is sensitive to the velocity distribution. Nevertheless our best model c predicts half of the unseen mass in M13 which is consistent with Lupton et al.'s (1987) result. Moreover similar results was obtained by Leonard et al. (1992). From the deep ccd photometry they suggest that half of the cluster mass may be in the form of low-mass stars and brown dwarfs. However the best results can be achieved by combining star count data and velocity data in maximum likelihood method. The possibility of that combination is already tested and the analysis will be presented in our future work.

ACKNOWLEDGEMENTS

The author would like to thank Dr. K. M. Cudworth for sending M13 proper motion data in computer file form.

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