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Evaluation of an Efficient Approximation to Many-on-Many Stochastic Combats

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Abstract

A time-varying nonhomogeneous Poisson process approximation of the nonexponential stochastic Lanchester model is defined and evaluated over a range of combat parameters including initial force sizes, breakpoints, and interkilling random variables. The proposed approximation is far excellent and takes much less CPU time than the existing models. The sensitivity analysis was performed to evaluate the efficiency of the proposed model and three recommended factors are suggested to guide the combat operators.

1. Introduction

It is widely known that the classical combat models such as the Deterministic Lanchester (DL) models and the Exponential Lanchester (EL) models do not represent the most realistic combat realization. The actual process in real systems can reasonably be expected to exhibit nonstationary phenomena. The methods of analytically representing nonstationary behavior of stochastic combat systems are not as available as steady state analysis methods. Simulation or numerical methods of analysis may be available for some nonstationary systems. The reason for this is that

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each combatant follows an ordinary terminating renewal process associated with him so that the entire combat becomes a combination of each combatant's process. Analytical methods of representing nonstationary behavior of stochastic combat situation are not as simple as stationary or deterministic systems. From the previous works of Ancker[1], Gafarian and Ancker[8], Gafarian and Manion[9], and Hong [13], we came to the conclusion that it is getting too complex to manipulate as the initial number of combatants is increased. The number of states in stochastic combat system increases exponentially for increasingly complex systems. Other methods of overcoming these defficulties are needed to provide at least good approximations for large combat size if not same as the Stochastic Lanchester (SL) model. Nonhomogeneous poisson processes of individual combatant may produce a better approximation.

Harvey[11] tried to investigate the validity of Nonhomogeneous Poisson processes Approximation (NPPA) as an alternative model that may describe combat situation very close to the real situation. He developed two different versions of NPPA computer simulation models which are called interfiring model and interkilling model. Monte-Carlo computer simualtion as an analysis method requires a significant amount of computer time. Output data from simulation often requires a significant amount of statistical analysis and a large number of replication would require to achieve small confidence intervals for mean measures of effectiveness. This study will consider the analytical way of solving the Nonhomogeneous poisson processes combat system.

2. Underlying Theories for the Stochastic Combat Models

2.1. Nonhomogeneous Poisson Process

Nonhomogeneous (nonstationary) poisson process is a generalization of the poisson process in which the property of stationary increments is dropped. The process is characterized by the following description found in Heyman and Sobel[12].

$\{N(t);\ t \geq 0\}$ is a counting process that is (W.p. 1) finite for finite values of t, $N(0)=0$, and has the following properties:

(1) Time-dependent increments:

$$\lim_{\Delta t \to 0} \frac{P\{N(t+\Delta t)-N(t)>0\}}{\Delta t} = \lambda(t)$$

(2) Independent increments: For any $t,\ s \geq 0$, $N(t+s)-N(t)$ is independent of $\{N(u);\ u \leq t\}$

(3) Orderliness: The jumps of N(t) are (w.p.1) of unit magnitude then $\{N(t);\ t \geq 0\}$ is a nonhomogeneous Poisson process with some finite rate $\lambda(t) \geq 0$.

If we neglect terms of size $o(\Delta t)$, then $\lambda(t)\Delta t$ is the probability of a jump during $(t,\ t+\Delta t)$. And Cox and Isham[6] state that "the important features of the process are that if $t_1, t_2, \cdots, t_k$ are arbitrary disjoint sets then $N(t_1), \cdots, N(t_k)$ are still independent poisson vbariables but with

$$E[N(ti)] = \int_{t_1} \lambda(t)dt, \quad i=1, \cdots, k.$$

Now the intervals between points of the process are not independent; further, the nonstationarity of the process complicates their distributions. "They also show the probability density function *(pdf)* of the interval x which starts at a point t_0 for a Nonhomogeneous poisson process to be,

$$f_x(x;t_0) = \lambda(t_0+x) \quad \exp\{-\int_{t_0}^{t_0+x} \lambda(t)dt\}. \tag{1}$$

with mean

$$E(x;t_0) = \int_0^\infty dx \quad \exp\{-\int_{t_0}^{t_0+x} \lambda(t)dt\}.$$

The cumulative distribution function *(cdf)* can be shown be

$$F_x(x;t_0) = 1 - \exp\{-\int_{t_0}^{t_0+x} \lambda(t)dt\}. \tag{2}$$

2.2 Renewal Process

Define the expected number of renewals by time t as $M(t) = E[N(t)]$, customary called the renewal function, and the corresponding rate function, called the renewal intensity function, is $m(t)$. An alternative definition of the renewal intensity function is that

$$m(t) = \lim_{\Delta t \to 0+} \frac{[P\{\text{one, or more renewals in}(t,\ t+\Delta t)\}]}{\Delta t}$$

or, alternatively, for a large number, N, of renewal process $Nm(t)\Delta t$ is the expected number of renewals in the time interval $(t,\ t+\Delta t)$. Also, the probability of more than one occurrence in Δt is $0(\Delta t)$.

2.2.1 backward Recurrence Time

The backward recurrence time, $Y(t)$, is defined to be the age of the combatant at time t. Cox [5] gives the pdf of $Y(t)$ as

$$f_{Y(t)}(x) = [1-F(t)]\delta(t-x) + m(t-x)[1-F(x)], \quad 0 \le x \le t$$

where $\delta(t-x) = 1$ if the first renewal time x exceeds t, and $\delta(t-x) = 0$, otherwise. And the cdf as

$$F_{Y(t)}(t) = \int_{t-y(t)}^{t} F^c(t-u)m(u)du, \quad 0 \le x \le t$$

where, $F(t)$ is the cdf of the interfiring time. The expected value of the hazard function is defined to be

$$E[r[Y(t)]] = \int_0^t r[Y(x)]f_{Y(t)}(x)dx. \qquad (3)$$

Substituting $r(t) = f(t)/F^c(t)$ into (3) yields

$$E[r[Y(t)]] = \int_0^t \frac{f(x)}{F^c(x)} f_{Y(t)}(x)dx. \qquad (4)$$

Substituting (4) into (3) yields

$$E[r[Y(t)]] = \int_0^t \frac{f(x)}{F^c(x)} [1-F(t)]\delta(t-X)m(t)X)[1-F(X)]dx.$$

Simplifying yields

$$E[r(t)] = f(t) + \int_0^t f(X)m(t-X)dx.$$

Which is well known renewal intensity function $m(t)$. Therefore, $m(t) = E[r(t)]$.

2.3 Superposition of Nonhomogeneous Poisson Preocesses

consider the superposition of L particular renewal processes at some time t.

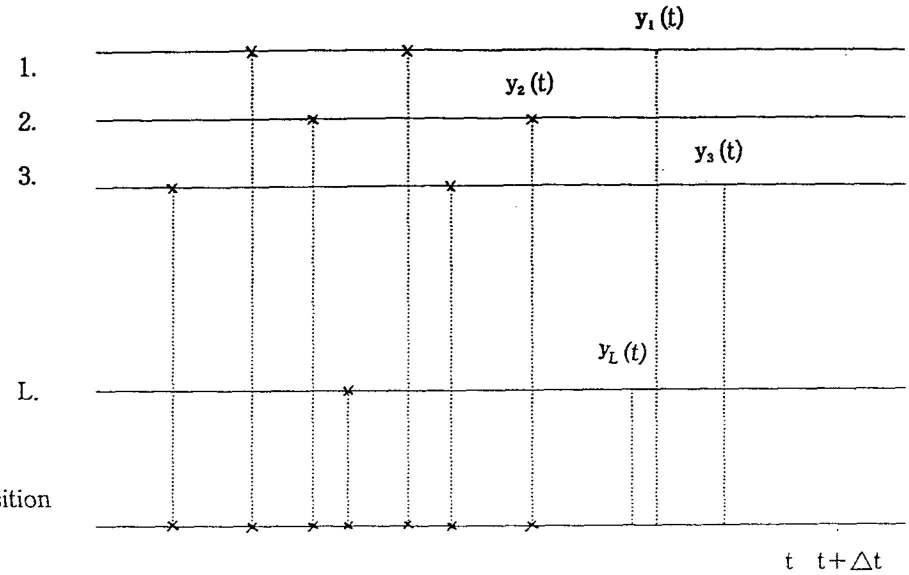

Figure-1 The Superposition of L Renewal Processes

Statement 1

let y_i (t) be the backward recurrence time of the i^{th} process at time t. Then the following statements can be obtained. Then

$$P[N(t, t+\Delta t) = 1 \mid y_1(t), y_2(t), \cdots, y_L(t)] =$$

$$r[y_1(t)]\Delta t \prod_{i\neq 1}^{L}\{1-r[Y_i(t)]\Delta t\} +$$

$$r[y_2(t)]\Delta t \prod_{i\neq 2}^{L}\{1-r[Y_i(t)]\Delta t\} +$$

$$r[y_3(t)]\Delta t \prod_{i\neq 3}^{L}\{1-r[Y_i(t)]\Delta t\} +$$

$$\vdots$$

$$r[y_L(t)]\Delta t \prod_{i\neq L}^{L}\{1-r[Y_i(t)]\Delta t\} .$$

Summing the terms and aggregating the second and higher order terms yields

$$\sum_{i=1}^{L} r[y_i(t)]\Delta t + 0(\Delta t).$$

Therefore,

$$P[N(t,t+\Delta t) = 1 \mid y_1(t),\ y_2(t),\ \cdots\ y_L(t)] = \sum_{i=1}^{L} r[y_i(t)]\Delta t + 0(\Delta t),$$

$$P[N(t,t+\Delta t) = 0 \mid y_1(t),\ y_2(t),\ \cdots\ y_L(t)] = 1-\sum_{i=1}^{L} r[y_i(t)]\Delta t + 0(\Delta t),$$

$$P[N(t,t+\Delta t) = 2 \mid y_1(t),\ y_2(t),\ \cdots\ y_L(t)] = 0(\Delta t), \qquad (5)$$

It is important to note that the above result depends upon the specific sample being observed. Dividing and nultiplying the sum in (5) gives the following :

$$P[N(t,t+\Delta t) = 1 \mid y_1(t),\ y_2(t),\ \cdots\ y_L(t)] = L\overline{r_1(t)}\ \Delta t + 0(\Delta t),$$

$$P[N(t,t+\Delta t) = 2 \mid y_1(t),\ y_2(t),\ \cdots\ y_L(t)] = 1-L\overline{r_1(t)}\ \Delta t + 0(\Delta t),$$

$$P[N(t,t+\Delta t) = 0 \mid y_1(t),\ y_2(t),\ \cdots\ y_L(t)] = 0(\Delta t)\ ,$$

Where

$$\overline{r_L(t)} = \frac{\sum_{i=1}^{L} r[y_i(t)]}{L}$$

If L is large enough, so that $\overline{r_L(t)} \sim E[r[Y(t)]]$ we would have :

$$P[N(t,t+\Delta t) = 1 \mid y_1(t),\ y_2(t),\ \cdots\ y_L(t)] = LE[r[Y(t)]]\Delta t + 0(\Delta t),$$

$$P[N(t,t+\Delta t) = 2 \mid y_1(t),\ y_2(t),\ \cdots\ y_L(t)] = 1-LE[r[Y(t)]]\Delta t + 0(\Delta t),$$

$$P[N(t,t+\Delta t) = 0 \mid y_1(t),\ y_2(t),\ \cdots\ y_L(t)] = 0(\Delta t)\ .$$

Statement 2

It is shown that $E[r[Y(t)]] = m(t)$. To show this in a more intuitive way consider the following.

$$M(t+\Delta t)-M(t) = 0P_0\ (t,t+\Delta t) + 1P_1\ (t,t+\Delta t) + \sum_{j=2}^{\infty} jp_j\ (t,t+\Delta t) \qquad (6)$$

where, $M(t) = E[N(t)]$, $N(t)$ = Number of renewals in $(0,\ t]$, and $P_j\ (t,t+\Delta t) = P[N(t,\ t+\Delta t)-N(t)=j]$, (i.e., the probability of exactly j renewals in $(t,t+\Delta t)$.

If the component renewal process has the property of orderliness, we have $P_j\ (t,t+\Delta t) = 0(\Delta t)$, $j\geq 2$, so that (6) becomes $M(t+\Delta t)-M(t) = m(t)\Delta t+0(\Delta t) = P_1\ (t,t+$

$\Delta t) + 0(\Delta t)$, where $m(t) = dM(t)/dt$.

Now we compute the P_1 (t, t+Δt) as the conditional expectation

$$p_1(t,\ t+\Delta t) = \int_0^t P[N(t+\Delta t) - N(t) = 1 \mid Y(t) = y]\ f_y(y)\,dy. \quad (7)$$

However,

$$P[N(t+\Delta t) - N(t) = 1 \mid Y(t) = Y] = r[Y(t)]\Delta t + 0(\Delta t; y). \quad (8)$$

So, substituting (8) into (7) yields

$$p_1\ (t, t+\Delta t) = E[r[Y(t)]]\Delta t + \int_0^t 0(\Delta t; y)\ f_y(y)\,dy.$$

The integral, by the mean value theorem, is

$$\int_0^t 0(\Delta t; y)\ f_{Y(t)}(y)\,dy = 0(\Delta t; \alpha t)t = 0(\Delta t),$$

Where, $0 \le \alpha \le 1$.

The minimum forward recurrence time, Y, of L i.i.d. Nonhomogeneous Poisson forward recurrence processes times, X_i, can be shown to be $P[Y = \min(X_1, X_2, \cdots, X_L) \le y] = P[Y \le y] = F_Y(y)$, which is, due to the i.i.d. of the X_i 's

$$\prod_{i=1}^{L} P[Y_i(t) > y] = P[Y(t) > y] = 1 - F_{Y(t)}(y).$$

Hence,

$$1 - F_Y(y) = \prod_{i=1}^{L}[1 - F_{x_i}(y)].$$

Again, because of the i.i.d. of the X_i's, for all i

$$F_{x_i}(y) = F_X(y)\ .$$

Therefore, $1 - F_Y(y) = [1 - F_X(y)]^L$. Simplifying and rearranging terms yields, $F_Y(y) = [1 - F_X(y)]^L$. By differentiating this with respect to y, We get

$$f_Y(y) = L[1 - F_X(y)]^{L-1} f_x(y)\ . \quad (9)$$

Since the Nonhomogeneous Poisson process depends upon the time t, substituting (1) and (2) for the pdf and cdf into (9), the pdf of the minimum event time becomes

$$f_Y\ (y;t) = L[1 - (1 - \exp\{-\int_t^{t+y} m(u)\,du\})]^{L-1} m(t+y) \exp\{-\int_t^{t+y} m(u)\,du\}$$

Simplifying yields,

$$f_Y\ (y;t) = Lm(t+y)\exp\{-\int_t^{t+y} Lm(u)\,du\}\ .$$

Therefore, the minimum interfiring time of L Nonhomogeneous Poisson processes at time t may be comsidered by using the following the renewal intensity function *Lm(t)*, where, *m(t)*=renewal intensity function evaluated at t. In a similar manner the minimum interkill time of L Nonhomogeneous Poisson processes at time t may be obtained by using the process with the following renewal intensity function LP_K m(t), where P_K = probability of kill.

3. Analytical Development of Nonhomogeneous Poisson Approximation

The Nonhomogeneous Poisson Processes model is significantly different from the stochastic model in the execution of the combat 8realization. The individual combatant's process is no longer considered, however, the superposed whole process will be modeled. Assuming the Lanchester square law of a combat realization as Dolansky [7] Listed :

(1) Two sides, homogeneous forces for each side, are engaged in a combat The rate of attrition may be different for each side.
(2) Each unit on either side is within weapon range of all units of the other side.
(3) Attrition-rate coefficients are constant.
(4) Each unit is informed about the location of the remaining opposing units so that when a target is destroyed, fire may be immediately shifted to a new target.
(5) Fire is uniformly distributed over remaining units.

Let us define the notation we will throughout the remainder of this study as follows :

$a_o(b_o)$: the initial number on side A(B) at time 0,

$a_f(b_f)$: breakpoint for side A(B), i.e., the number of side A(B) at time the A(B) side loses,

A(t) : number of survivors on side A at time t,

B(t) : number of survivors on side B at time t,

$P_{ab}(t)$: $P[A(t)=a,\ B(t)=b]$, a state probability function,

$m_A(t)$: $E[A(t)]$, expected value of $A(t)$,

$m_B(t)$: $E[B(t)]$, expected value of $B(t)$,

$\delta_A(t)$: standard deviation of $A(t)$,

$\delta_B(t)$: standard deviation of $B(t)$,

$P[i]$: probability i side win, i=A or B,

T_D : random variable, time duration of combat,

$F^c_{T_D}(t)$: complementary distribution function of T_D,

μ_{T_D} : expected value of T_D,

δ_{T_D} : Standard deviation of T_D,

$P_A(P_B)$: constant single shot kill probability of a combatant on side A(or B).

$R_A(R_B)$: $P_A/\mu_B(P_B/\mu_B)$, a combatant's kill rate on side A(or B).

$r_A(r_B)$: instantaneous kill rate at time t for a combatant's on side A(or B).

We need to introduce interfiring intensity functions, $\lambda_A(t)$ for side A and $\lambda_B(t)$ for side B, in this model. Hence the intensity functions of interkilling process will be $P_A\ \lambda_A(t)$ for side A and $P_B\ \lambda_B(t)$ for side B. There are initially a_o combatants on side A and b_o on the side B. The battle is continued until either side A or side B has reached its specified breakpoint (a_f or b_f) .

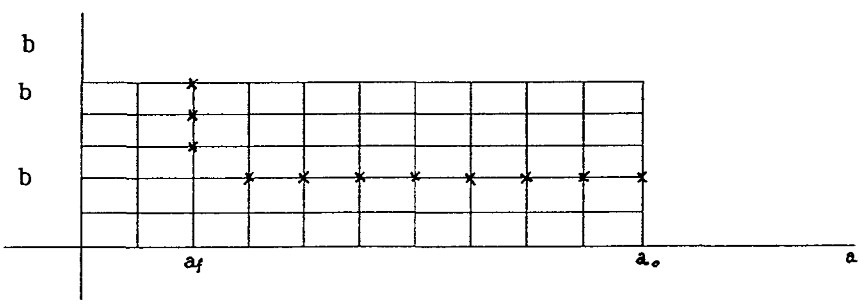

Figure-2 : Discrete State Space

Now we will turn to the analytical development of the Kolmogorov forward equations, Which shows all the probabilistic evolution of the state of each side. A state probability P_{ab} (t) is defined as the probability of being at a general lattice point (a, b) shown in Figure-2 at time t.

The battle terminatiom point marked as * in the Figure-2 are called absorbing states and all other states are called transient states. The state (a_f , b_f) can never be reached. The characteristics of state probabilities are

- Initial condition $P_{a_o b_o}(o) = 1$
- Transient states $P_{ab} \rightarrow 0$ as time $t \rightarrow \infty$ where $a = a_o, a_o-1, a_o-2, \cdots a_f+1$ and $b = b_o, b_o-1, b_o-2, \cdots, b_f+1$.
- Absorbing states $P_{a_f b}(t)$ (or $P_{ab_f}(t)$) > 0 Where $a = a_o, a_o-1, a_o-2, \cdots a_f+1$ and $b = b_o, b_o-1, b_o-2, \cdots, b_f+1$.
- Therefore, sum of the absorbing states $\sum_{a=a_f+1}^{a_o} P_{ab_f}(t) + \sum_{b=b_f+1}^{b_o} P_{a_f b}(t) \rightarrow 1$, as time $t \rightarrow \infty$.

The standard approach for representing the transient and nonstationary behavior of this nonhomogeneous Poisson process is to numerically integrate the time differential difference equations representing the probabilities of being in each of the discrete states. This set of time derivatives are called the Kolmogorov forward equations and are commonly used to analyze time varying behavior of queues. For a (a_o , a_f ; b_o , b_f) nonhomogeneous Poisson process combat system the Kolmogorov forward equations are :

<u>for initial state</u>

$$P_{a_o b_o}(t+\Delta t) = P_{a_o b_o}(t)\{1-a_o P_A \lambda_A(t)\Delta t\}\{1-b_o P_B \lambda_B(t)\Delta t\} .$$

Rearranging terms, dividing by Δt, and letting $\Delta t \rightarrow 0$, we get

$$\frac{dP_{a_o b_o}(t)}{dt} = -P_{a_o b_o}(t)\{a_o P_o \lambda_A(t) + b_o P_B \lambda_B(t)\} .$$

and

$$P_{a_o b_o-1}(t+\Delta t) = P_{a_o b_o-1}(t)\{1-a_o P_A \lambda_A(t)\Delta t\}\{1-(b_o-1)P_B \lambda_B(t)\Delta t\}$$
$$+ P_{a_o b_o}(t)\, a_o P_A \lambda_A(t)\Delta t\{1-b_o P_B \lambda_B(t)\Delta t\} .$$

hence this equation yields

$$\frac{dP_{a_o b_o-1}(t)}{dt} = -P_{a_o b_o-1}(t)\{a_o P_A \lambda_A(t) + (b_o-1)P_B \lambda_B(t) + P_{a_o b_o}(t)\, a_o P_A \lambda_A(t)\} .$$

Similarly, for $a = a_o$ and $b_f < b < b_o - 1$

$$\frac{dP_{a_ob}(t)}{dt} = -P_{a_ob}(t)\{a_oP_A\lambda_A(t) + bP_B\lambda_B(t) + P_{a_ob+1}(t)a_oP_A\lambda_A(t)\} .$$

for $a_f < a < a_o$ and $b = b_o$

$$\frac{dP_{ab_o}(t)}{dt} = -P_{ab_o}(t)\{aP_A\lambda_A(t) + b_oP_B\lambda_B(t) + P_{a+1b_o}(t)b_oP_B\lambda_B(t)\} .$$

for $a_f < a < a_o$ and $b_f < b < b_o$

$$\frac{dP_{ab}(t)}{dt} = -P_{ab}(t)\{aP_A\lambda_A(t) + bP_B\lambda_B(t)\} + P_{a+1b}(t)bP_B\lambda_B(t)$$
$$+ P_{ab+1}(t)aP_A\lambda_A(t) ,$$

for $a = a_f$ and $b_f < b \le b_o$

$$\frac{dP_{a_fb}(t)}{dt} = -P_{a_f+1b}(t)bP_B\lambda_B(t)\} .$$

for $a_f < a \le a_o$ and $b = b_f$

$$\frac{dP_{ab_f}(t)}{dt} = -P_{ab_f+1}(t)aP_A\lambda_A(t)\} .$$

with initial conditions $P_{a_ob_o}(0) = 1$ and $P_{ab}(0) = 0$ for all absorbing states and transient states.

The transient state probabilities are obtained by solving a system of Kolmogorov forward equations described above. However, since nonstationary analysis is indeed mathematically intractable numerical methods have been applied in the literature. The Runge-Kutta method with step width control, due to Verner, is the basis of a very successful differential equation solving a subroutine named DVERK which is widely available in subroutine libraries(Hull, Enright, and Jackson[14]). Verner's method, which is called Runge-Kutta-Verner fifth-sixth order, requires eight function evaluations per step, and from these, two estimated values of function are obtained, one based on a *fifth order* and another based on sixth order approximation. The method was incorporated into the subroutine DVERK and disseminated by IMSL [15] Inc., Houston, Texas.

3. 1 Combat Figures of Merit

The state probabilities produce essentially all the information in a combat operation :

(1) The means and variances of the survivors on both sides as a function of time t,

$$m_A(t) = E[A(t)] = \sum_{a=a_f}^{a_o}\sum_{b=b_f}^{b_o} aP_{ab}(T) \;,\; m_B(t) = E[B(T)] = \sum_{a=a_f}^{a_o}\sum_{b=b_f}^{b_o} bP_{ab}(t)$$

where $P_{a_f b_f}(t) = 0$ for any t. computing

$$E[A^2(t)] = \sum_{a=a_f}^{a_o}\sum_{b=b_f}^{b_o} a^2P_{ab}(t) \;,\; E[B^2(t)] = \sum_{a=a_f}^{a_o}\sum_{b=b_f}^{b_o} b^2 P_{ab}(t)$$

where again $P_{a_f b_f}(t) = 0$ for any t, so we can have

$$\delta_A(t) = \sqrt{E[A^2(t)]-m_A^2(t)} \;,\; \delta_B(t) = \sqrt{E[B^2(t)]-m_B^2(t)} \;.$$

(2) The expected value and standard deviation of T_D , the battle termination time, are computed with commonly used integral equations

$$E[T_D] = \int_0^\infty F_{T_D}^c(t)\, dt, \quad E[TD^2] = 2\int_0^\infty t\, F_{T_D}^c(t)\, dt,$$

Where

$$F_{T_D}^c(t) = \sum_{a=a_f+1}^{a_o} \sum_{b=b_f+1}^{b_o} P_{ab}(t) \;.$$

Thus

$$E[T_D^2] = 2\int_0^\infty t \sum_{a=a_f+1}^{a_o} \sum_{b=b_f+1}^{b_o} P_{ab}(t) \;.$$

(3) the probabilities that A wins, P[A], and B wins, P[B], are

$$P[A] = \lim_{t\to\infty} \sum_{a=a_f+1}^{a_o} P_{ab_f}(t) \quad P[B] = 1-P[A] = \lim_{t\to\infty} \sum_{b=b_f+1}^{b_o} P_{a_f b}(t)$$

4. Results and Comparisons

The NPPA model developed in this study needs to have the appropriate interkilling intensity functions for both sides to generate the interkilling times. The renewal density functions is used for solving the Kolmogorov forward equations. The numerical inversion of the Laplace transform has been applied to get $\lambda_A(A)$ and $\lambda_B(t)$ (references [10, 13])

The verification of the model is done in two phases. First, all the transient state and absorbing state probabilities are observed at any time t. The transient state probabilities have to converge to zero at time infinity and the absorbing state probabilities should have nonzero probabilities at the end of the battle. Furthermore, the sum of all the state probabilities must be equal to one at any time. The second procedure of model verification involves comparing the output of the model with simulation results provided by Gafarian, Harvey, Hong, and Kronauer [10] and Harvey [11]. Simulations were run for one million replications to ensure that the estimate of all eight overall combat figures of merit were reliable. The relative difference between the NPPA model and simulation output is used as a measure of the NPPA model's veracity. Table-1 presents the values of the relative difference for the overall battle measures between NPPA and simulation output. In both cases, the relative errors of all eight parameters are in between 0% and .20%.

	($a_o=2$, $a_f=0$, $b_o=2$, $b_f=0$)			($a_o=5$, $a_f=4$, $b_o=5$, $b_f=4$)		
Figures of Merit	True Value	Simulation Estimate	Relative Error %	True Value	Simulation Estimate	Relative Error %
$E[T_D]$	9.1750	9.1899	-.162	1.4666	1.4664	.014
$\sigma[T_D]$	6.6750	6.6840	-.135	1.2262	1.2262	.399
$E[A(\infty)]$	1.1059	1.1057	.018	4.6188	4.6188	.000
$\sigma[A(\infty)]$	.8927	.8926	.011	.4857	.4857	.000
$E[B(\infty)]$	.5744	.5744	.000	4.3812	4.3812	.000
$\sigma[B(\infty)]$	.8312	.8313	-.012	.4857	.4857	.000
$P[A]$	.6489	.6489	.000	.6188	.6188	.000
$P[B]$	.3511	.3511	.000	.3812	.3812	.000

Table-1 Verification results of Erlang-2 interfiring time for both sides; $P_A=.10$, $\mu_A=1.00$, $P_B=1.50$, $\mu_B=1.50$, and 1,000,000 replications for simulation output.

To see how the approximation behaves when the time comes and to inplement the NPPA model the several combats with different size of combatants were examined. Three main factors come up with the efficiency of this NPPA model. These three factors are :

Factor 1 The sizes of breakpoint for both sides.

A better approximation can be achieved when the breakpoints (a_f and b_f) are near the initial number of combatants (a_0 of b_0) . The superposed process may be getting away from a nonhomogeneous poisson process due to the less number of remaining combatants on both sides as time goes on. Table-2 shows the mean and maximum relative differences between three models (stochastic Lanchester (SL) versus NPPA and Stochastic Lanchester (SL) versus Exponential Lanchester (EL)) for eight overall combat measures. The mean relative differences are decreased as we expected when the both initial and breakpoint are increased. However, it is noted that there is a significant difference between SL and EL although the battle sizes are large. This says something about the EL model as some researchers already have mentioned. (references [2], [9], [10], [13])

(a_0, a_f) (b_0, b_f)	SL versus NPPA	SL versus EL
(10, 0) (10, 0)	(Mean) 3.80 % (Maximum) (7.60)	18.50 % (44.38)
(20, 10) (20, 10)	3.14 (7.53)	19.70 (50.41)
(30, 20) (30, 20)	2.78 (6.60)	25.29 (72.56)
(40, 30) (40, 30)	2.35 (5.93)	30.01 (89.74)
(50, 40) (50, 40)	2.07 (4.63)	33.66 (104.76)

Table-2 Relative differences between three models (SL, EL, and NPPA) ; Erlang-2 interfiring random variable on both sides; $\mu_A = 1.00$, $P_A = .50$, $\mu_B = 1.80$, $P_B = .90$.

Factor 2 Total number of states.

The total number of states is $(a_0\text{-}a_f+1)(b_0\text{-}b_f+1)\text{-}1$. For the fixed number of breakpoints on both sides a_0 and b_0 are inportant elements to obtain a good approximation. As we increase the initial number of combatants on both sides the relative differences are getting larger. Table-3 shows this phenomena for four different set of combat sizes.

(a_o, a_f) (b_o, b_f)	SL versus NPPA	SL versus EL
(15, 10) (15, 10)	(Mean) 3.20 % (Maximum) (9.37)	17.94 % (43.24)
(20, 10) (20, 10)	3.14 (7.53)	19.70 (50.41
(30, 10) (30, 10)	4.42 (4.42)	24.72 (67.79)
(40, 10) (40, 10)	4.64 (10.74)	29.08 (83.76)

Table-3 Relative differences between three models (SL, EL, and NPPA); Erlang-2 interfiring random variable on both sides; $\mu_A = 1.00$, $P_A = .50$, $\mu_B = 1.80$, $P_B = .90$.

Factor 3 Battle duration

For a given set of combat sizes the NPPA model produces a better approximation as the battle termination time is getting longer since the battle is conducted with more combatants on both sides. Table-4 presents the effectiveness of the battle duration.

(μ_A, P_A) (μ_B, P_B)	SL versus NPPA	SL versus EL
(1.00, .10) (1.80, .18)	(Mean) .81 % (Maximum) (2.60)	5.17 % (8.92)
(1.00, .30) (1.80, .54)	2.10 (5.79)	13.14 (29.22)
1.00, .50) (1.80, .90)	3.14 (7.53)	19.70 (50.41)

Table-4 Relative differences between three models (SL, EL, and NPPA); Erlang-2 interfiring random variable on both sides; $(a_o = 20, a_f = 10)$ and $b_o = 20$, $b_f = 10)$.

Finally, the NPPA model requires much less CPU time than the actual SL model or EL model. Table-5 contains some computation times on VAX/VMS system. For the actual SL model we ran one million replications on each battle to get precise output values. It was found that the NPPA is fast and produces very accurate results.

(a_0, a_f) (b_0, b_f)	SL Model Hrs : Min : Sec	NPPA Model Hrs : Min : Sec
(10, 0) (10, 0)	11 : 16 : 08	00 : 05 : 34
(20, 10) (20, 10)	00 : 24 : 55	00 : 05 : 37
(30, 20) (30, 20)	01 : 16 : 19	00 : 10 : 22
(40, 30) (40, 30)	02 : 03 : 19	00 : 16 : 18
(50, 40) (50, 40)	04 : 09 : 31	00 : 22 : 19

Table-5 Comparison for CPU times between two models : the SL results are based on the simulation of 1,000,000 replications. $\mu_A = 1.00$, $P_A = .10$, $\mu_B = 1.80 P_B = .50$ and Erlang-2 interfiring time random variables.

5. Conclusions

The main view of the research was the development of the efficient numerical approximation to the many-on-many stochastic square law combat model. The nonhomogeneous Poisson process is considered as an alternative method for the approximation. Given any arbitrary distribution for the interfiring time random variables for both sides the exact interkilling renewal intensity is applied for solving the system of Kolmogorov forward equations. The Runge-Kutts method is used for solving these equations to obtain the state probabilities simultaneously. The efficiency turned out to be very excellent and computer time could be saved when it was compared with the simulation model. There are three factors are observed in the proposed NPPA model. First, a better approximation can be achieved when the breakpoints are near the initial number of combatants. Second, for the fixed number of breakpoints on both sides the initial sizes of combatant, a_0 and b_0 , are inportant elements to obtain a good approximation. And third, for a given set of combat sizes the NPPA model produces a better approximation as the battle termination time is getting longer since the battle is conducted with more combatants on both sides. Some experimental results show that it is very important for the combat operators to note the differences between actual Stochastic Lanchester (SL) model and the traditional Exponential Lanchester(EL) model. Furthermore, the Deterministic Lanchester(DL) model should not be considered any more. The results of many previous works support this statement. The NPPA model may be replaced with SL model because it is extremely difficult to find the solution analytically and is time consuming to simulate it.

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